Junior Mathematical Challenge

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Solutions and investigations

These solutions augment the shorter solutions also available online. The solutions given here are full solutions, as explained below. In some cases we give alternative solutions. There are also many additional problems for further investigation. We welcome comments on these solutions. Please send them to enquiry@ukmt.org.uk.

The Junior Mathematical Challenge (JMC) is a multiple-choice paper. For each question, you are presented with five options, of which just one is correct. It follows that occasionally you can find the correct answers by working backwards from the given alternatives, or by showing that four of them are not correct. This can sometimes be a sensible thing to do in the context of the JMC.

However, this does not provide a full mathematical explanation that would be acceptable if you were just given the question without any alternative answers. Therefore here we have aimed at giving full solutions with all steps explained (or, occasionally, left as an exercise). We hope that these solutions can be used as a model for the type of written solution that is expected when a complete solution to a mathematical problem is required (for example, in the Junior Mathematical Olympiad and similar competitions).

These solutions may be used freely within your school or college. You may, without further permission, post these solutions on a website that is accessible only to staff and students of the school or college, print out and distribute copies within the school or college, and use them in the classroom. If you wish to use them in any other way, please consult us.

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1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
E	D	C	B	E	C	A	D	A	D	E	B	B	D	E	A	E	B	E	D	D	E	B	B	E

1. Exactly one of the following five numbers is *not* prime. Which is it?

A 101 B 103 C 107 D 109 E 111

SOLUTION **E**

COMMENTARY

An integer n, with $n > 1$, is not a prime if it has a factor which is greater than 1 but less than n. So we need to decide which of the given options has a factor of this kind.

We see that all the options are odd numbers. So none of them has 2 as a factor. We next look to see if any of them has 3 as a factor. This leads us to the option that is not a prime.

In the context of the JMC it is adequate to spot the one number that is not prime. For a full solution it is necessary to check that all of the other options are primes. You are asked to do this in Problem 1.2.

METHOD 1

$111 = 3 \times 37$. Therefore 111 has 3 as a factor and so is not prime.

METHOD 2

The test for whether an integer is a multiple of 3 is that the sum of its digits is a multiple of 3. We therefore check the sums of the digits of the numbers given as options. We obtain

$$1 + 0 + 1 = 2; \quad 1 + 0 + 3 = 4; \quad 1 + 0 + 7 = 8; \quad 1 + 0 + 9 = 10; \quad 1 + 1 + 1 = 3.$$

Thus we see that sum of the digits of 111 is a multiple of 3. Hence 111 is a multiple of 3. Therefore 111 is not prime.

FOR INVESTIGATION

1.1 At first sight it may seem that to check whether a positive integer n, with $n > 1$, is prime, we need to check whether it is divisible by each of the integers k in the range $2 \leq k < n$.

It turns out, however, that we can cut down the amount of work involved. This is because it is only necessary to check for divisibility by the primes p in the range $2 \leq p \leq \sqrt{n}$.

Explain why.

1.2 (a) Check that 101, 103, 107 and 109 are all prime.

(b) Show that none of 201, 203, 207 and 209 is prime.

1.3 Show that the test for whether an integer is a multiple of 3, as used in Method 2, is correct.

1.4 Count the number of primes in each of the ranges $90 \leq n \leq 99$, $100 \leq n \leq 109$, and $110 \leq n \leq 119$.

2. What is the value of $2020 \div 20$?

A 10 B 11 C 100 D 101 E 111

SOLUTION **D**

METHOD 1

We have

$$2020 \div 20 = \frac{2020}{20} = \frac{202 \times 10}{2 \times 10} = \frac{202}{2} = \frac{2 \times 101}{2} = 101.$$

METHOD 2

We have

$$\begin{aligned} 2020 &= 2000 + 20 \\ &= 20 \times 100 + 20 \times 1 \\ &= 20 \times (100 + 1) \\ &= 20 \times 101. \end{aligned}$$

Therefore

$$2020 \div 20 = 101.$$

FOR INVESTIGATION

2.1 What are the answers to the following division sums?

(a) $202\,020 \div 20$,

(b) $3737 \div 37$,

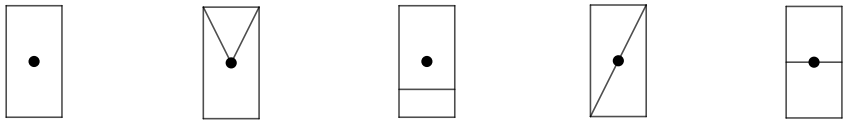
(c) $789\,789 \div 789$.

2.2 Express 2020 as the product of prime numbers.

2.3 You are given that $202\,020 \div n = 60$. Find the value of n.

2.4 Express $20\,202\,020$ as the product of prime numbers.

3. Each of these figures is based on a rectangle whose centre is shown.



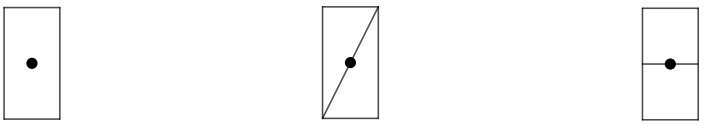
How many of the figures have rotational symmetry of order two?

A 1 B 2 C 3 D 4 E 5

SOLUTION **C**

A rotation of order two is a half turn. Therefore we need to decide which of these figures looks the same when it is rotated through 180° about its centre.

We see that the figures



look the same when rotated through 180°, but the other two do not.

Therefore the number of the figures that have rotational symmetry of order two is 3.

FOR INVESTIGATION

3.1 How many reflectional symmetries does each of the figures in this question have?

3.2 Give an example of a figure with a rotational symmetry of order three.

3.3 Give an example of a figure with a rotational symmetry of order four.

3.4 Is it possible to have a figure with a rotational symmetry of order four, but not a rotational symmetry of order two?

4. How many centimetres are there in 66.6 metres?

A 66600 B 6660 C 666 D 66.6 E 66

SOLUTION **B**

There are 100 centimetres in one metre.

Therefore in 66.6 metres there are 66.6×100 centimetres.

Multiplying by 100 is effected by moving the decimal point two places to the right (or, equivalently, by moving the digits two places to the left). Therefore $66.6 \times 100 = 6660$.

Hence there are 6660 centimetres in 66.6 metres.

FOR INVESTIGATION

4.1 How many centimetres are there in 66.6 kilometres?

5. Amrita thinks of a number. She doubles it, adds 9, divides her answer by 3 and finally subtracts 1. She obtains the same number she originally thought of.
What was Amrita's number?

A 1 B 2 C 3 D 4 E 6

SOLUTION **E**

METHOD 1

We check each option in turn to see which would leave Amrita with the number she began with. We obtain

initial number	double	add 9	divide by 3	subtract 1
1	2	11	$\frac{11}{3}$	$\frac{8}{3}$
2	4	13	$\frac{13}{3}$	$\frac{10}{3}$
3	6	15	5	4
4	8	17	$\frac{17}{3}$	$\frac{14}{3}$
6	12	21	7	6

We see that, of the given options, the only number Amrita can start with and obtain the same number at the end is 6.

METHOD 2

In this method we save work by using algebra.

Suppose Amrita thinks of the number x. She doubles it to obtain $2x$, then adds 9 to obtain $2x + 9$, divides by 3 to obtain $\dfrac{2x+9}{3}$, and finally subtracts 1 to end up with $\dfrac{2x+9}{3} - 1$.

Because Amrita ends up with the number she first thought of

$$\frac{2x+9}{3} - 1 = x.$$

If we multiply both sides of this equation by 3, we obtain

$$2x + 9 - 3 = 3x.$$

This equation may be rearranged as $9 - 3 = 3x - 2x$, which gives $x = 6$.

FOR INVESTIGATION

5.1 Amrita thinks of a number, trebles it, adds 16, and divides the answer by 4. She then subtracts 2 and ends up with the number she first thought of. What was this number?

6. What is the value of $\frac{6}{12}-\frac{5}{12}+\frac{4}{12}-\frac{3}{12}+\frac{2}{12}-\frac{1}{12}$?

A $\frac{1}{2}$ B $\frac{1}{3}$ C $\frac{1}{4}$ D $\frac{1}{5}$ E $\frac{1}{6}$

SOLUTION **C**

We have

$$\begin{aligned}\frac{6}{12}-\frac{5}{12}+\frac{4}{12}-\frac{3}{12}+\frac{2}{12}-\frac{1}{12} &= \frac{6-5+4-3+2-1}{12}\\ &= \frac{(6-5)+(4-3)+(2-1)}{12}\\ &= \frac{1+1+1}{12}\\ &= \frac{3}{12}\\ &= \frac{1}{4}.\end{aligned}$$

FOR INVESTIGATION

6.1 What is the value of

$$\frac{8}{12}-\frac{7}{12}+\frac{6}{12}-\frac{5}{12}+\frac{4}{12}-\frac{3}{12}+\frac{2}{12}-\frac{1}{12}\,?$$

6.2 What is the value of

$$\frac{100}{12}-\frac{99}{12}+\frac{98}{12}-\frac{97}{12}+\cdots+\frac{4}{12}-\frac{3}{12}+\frac{2}{12}-\frac{1}{12}\,?$$

Note:

The dots in the above expression mean that + and − signs alternate all the way down until the fractions $+\frac{2}{12}$ and $-\frac{1}{12}$ are reached.

7. Four different positive integers have a product of 110. What is the sum of the four integers?

A 19 B 22 C 24 D 25 E 28

SOLUTION **A**

The factorization of 110 into primes gives

$$110 = 2 \times 5 \times 11.$$

At first sight this suggests that we can find at most three positive integers whose product is 110. However, although 1 is not a prime factor of 110, we can use it to write

$$110 = 1 \times 2 \times 5 \times 11.$$

We leave it as an exercise (Problem 7.1) to explain why this is the only way to express 110 as the product of four different positive integers.

The sum of these four positive integers whose product is 110 is given by

$$1 + 2 + 5 + 11 = 19.$$

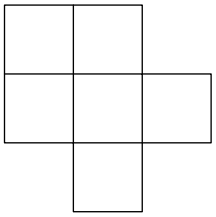
FOR INVESTIGATION

7.1 Explain why, if 110 is written as the product of four different positive integers, it follows that these positive integers can only be 1, 2, 5 and 11 (in some order).

7.2 Which is the smallest integer that is the product of four different positive integers, each greater than 1?

8. Wesley has a grid of six cells. He wants to colour two of the cells black so that the two black cells share a vertex but not a side. In how many ways can he achieve this?

A 2 B 3 C 4 D 5 E 6



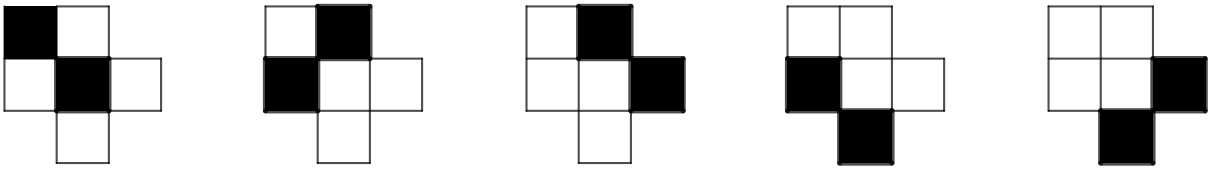
SOLUTION **D**

METHOD 1

The most straightforward method is to draw all the different ways in which Wesley could colour black two cells which share a vertex but not a side.

To avoid missing a case, we consider each square in turn, starting from the top left square, and see whether it can be the top square of two squares that share a vertex but not a side.

This leads to the following 5 arrangements.



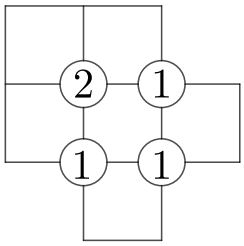
Note: This method would not work so well with a larger grid, when drawing all the different possibilities would be cumbersome.

METHOD 2

We work out for each vertex how many pairs of cells share that vertex but not a side. This number can only be 0, 1 or 2.

In the diagram we have indicated this number when it is 1 or 2.

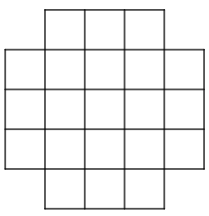
The total number of ways Wesley can colour two cells joined by a vertex but not by an edge is the sum of these numbers. We see that this sum is 2 + 1 + 1 + 1 = 5.



FOR INVESTIGATION

8.1 Wesley now has a grid of 21 cells as shown.

In how many ways can he colour two of the cells black so that the two black cells share a vertex, but not a side?



9. One half of one third of one quarter of one fifth of a number is 2.
What is the number?

A 240 B 120 C 60 D $\frac{1}{120}$ E $\frac{1}{240}$

SOLUTION **A**

Let the number we seek be x.

We have

$$\frac{1}{2} \times \left(\frac{1}{3} \times \left(\frac{1}{4} \times \left(\frac{1}{5} \times x\right)\right)\right) = 2.$$

This is equivalent to

$$\left(\frac{1}{2} \times \frac{1}{3} \times \frac{1}{4} \times \frac{1}{5}\right)x = 2,$$

that is,

$$\left(\frac{1}{2 \times 3 \times 4 \times 5}\right)x = 2.$$

This last equation is equivalent to

$$\frac{x}{120} = 2.$$

Hence, multiplying both sides by 120,

$$x = 120 \times 2 = 240.$$

Therefore the number we seek is 240.

FOR INVESTIGATION

9.1 One half of one third of one quarter of one fifth of one sixth of a number is 3.
What is the number?

9.2 One half of two thirds of three quarters of a number is 1.
What is the number?

9.3 One half of two thirds of three quarters of four fifths of a number is 1.
What is the number?

9.4 One half of two thirds of three quarters of four fifths of five sixths of a number is 7.
What is the number?

10. How many of these equations have the solution $x = 12$?

$$x - 2 = 10 \quad \tfrac{x}{2} = 24 \quad 10 - x = 2 \quad 2x - 1 = 25$$

A 4 B 3 C 2 D 1 E 0

SOLUTION **D**

When $x = 12$ we have

$$\begin{aligned} x - 2 &= 12 - 2 = 10, \\ \tfrac{x}{2} &= \tfrac{12}{2} = 6, \\ 10 - x &= 10 - 12 = -2, \\ \text{and} \quad 2x - 1 &= 2 \times 12 - 1 = 24 - 1 = 23. \end{aligned}$$

We therefore see that $x = 12$ is a solution of the first equation, but is not a solution of the other three equations. Thus just one of the equations has the solution $x = 12$.

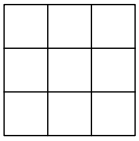
FOR INVESTIGATION

10.1 Find the solutions of the equations $\frac{x}{2} = 24$, $10 - x = 2$ and $2x - 1 = 25$.

10.2 Give examples of other equations that have $x = 12$ as a solution.

10.3 Give an example of a quadratic equation that has $x = 12$ as a solution.

11. This 3 by 3 grid shows nine 1 cm × 1 cm squares and uses 24 cm of wire. What length of wire is required for a similar 20 by 20 grid?



A 400 cm B 420 cm C 441 cm D 800 cm E 840 cm

SOLUTION **E**

A similar 20 by 20 grid uses 21 horizontal pieces of wire, each of length 20 cm, and 21 vertical pieces of wire, also each of length 20 cm.

Therefore the total length of wire needed for a 20 by 20 grid is

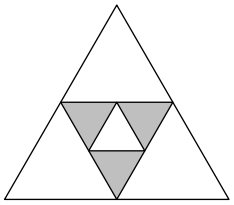
$$2 \times (21 \times 20)\,\text{cm} = 2 \times 420\,\text{cm} = 840\,\text{cm}.$$

FOR INVESTIGATION

11.1 A similar $n \times n$ grid uses 4900 cm of wire. What is the value of n?

12. The diagram shows an equilateral triangle divided into four smaller equilateral triangles. One of these triangles has itself been divided into four smaller equilateral triangles.
What fraction of the area of the large triangle has been shaded?

A $\frac{1}{8}$ B $\frac{3}{16}$ C $\frac{1}{4}$ D $\frac{5}{16}$ E $\frac{3}{8}$



SOLUTION **B**

The four smaller equilateral triangles into which the large equilateral triangle is divided share sides. So their side lengths are all the same. Therefore these four equilateral triangles have the same area.

It follows that the area of each of these triangles is one-quarter of the area of the large triangle.

Similarly, the area of each of the three shaded triangles is one-quarter of the area of the central triangle.

Therefore three-quarters of the central triangle is shaded.

Hence three-quarters of one-quarter of the large triangle is shaded.

It follows that the fraction of the area of the large triangle that is shaded is given by

$$\frac{3}{4} \times \frac{1}{4} = \frac{3}{16}.$$

FOR INVESTIGATION

12.1 Why does it follow from the fact that two equilateral triangles have the same side lengths that they have the same area?

13. The mean of four positive integers is 5. The median of the four integers is 6.
What is the mean of the largest and smallest of the integers?

A 3 B 4 C 5 D 6 E 8

SOLUTION **B**

COMMENTARY

To answer this question we need to know how to work out the *median* of a collection of numbers.

When we have an odd number of numbers, their median is the middle number when they are arranged in order of size.

For example to find the median of the five numbers 11, 5, 99, 17, 2, we first arrange

them in increasing order as

$$2,\ 5,\ 11,\ 17,\ 99.$$

The number in the middle in this ordering is 11. Hence the median of the numbers 11, 5, 99, 17, 2 is 11.

When we have an even number of numbers, their median is the number that is half way between the two middle numbers when they are arranged in order of size.

The number half way between two numbers is the same as their mean.

For example, to find the median of the six numbers 11, 120, 99, 17, 2, 233, we first arrange them in increasing order as

$$2,\ 11,\ 17,\ 99,\ 120,\ 233.$$

The two numbers in the middle in this ordering are 17 and 99. The mean of these two numbers is $\frac{1}{2}(17 + 99) = 58$. Hence the median of the numbers 11, 120, 99, 17, 2, 233 is 58.

Because the four positive integers have mean 5, their sum is 4×5, that is, 20.

Because the median of the four integers is 6, the mean of the second and third in order of size is 6. Hence their sum is 2×6, that is, 12.

It follows that the sum of the largest and smallest of the integers is $20 - 12$, that is, 8.

The mean of the largest and smallest of the integers is half their sum. So their mean is half of 8, that is, 4.

FOR INVESTIGATION

13.1 Give an example of a set of four different positive integers which has mean 5 and median 6.

13.2 Find all the different sets of four different positive integers which have mean 5 and median 6.

13.3 What is the median of the numbers 23, 4, 19, 5, 26, 11, 17?

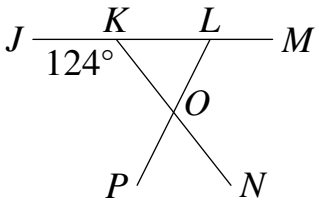
13.4 What is the median of the numbers 23, 4, 19, 5, 26, 11, 17, 21?

13.5 Give an example of a set of eight different numbers whose median is greater than their mean.

13.6 Give an example of a set of eight different numbers whose mean is greater than their median.

14. In the diagram, angle OLM is twice as large as angle PON. What is the size of angle OLM?

A 102° B 106° C 108° D 112°
E 124°



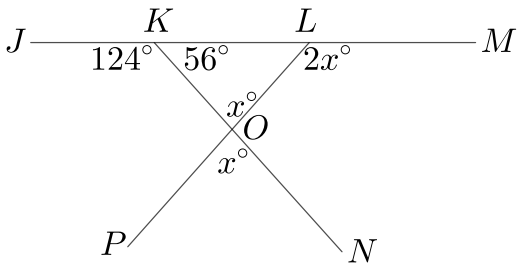
SOLUTION **D**

To avoid fractions, we let $\angle OLM = 2x°$.

Then, because $\angle OLM$ is twice $\angle PON$, we have $\angle PON = x°$.

Because the angles $\angle PON$ and $\angle KOL$ are vertically opposite, $\angle KOL = \angle PON = x°$.

Because angles on a line have sum 180°, it follows that $\angle OKL + \angle JKO = 180°$ and therefore $\angle OKL = 180° - 124° = 56°$.



By the External Angle Theorem, $\angle OLM = \angle KOL + \angle OKL$. Therefore

$$2x = x + 56.$$

It follows that

$$x = 56.$$

Hence $2x = 2 \times 56 = 112$.

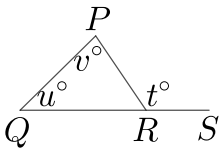
We conclude that $\angle OLM$ is 112°.

FOR INVESTIGATION

14.1 The *External Angle Theorem* says that the external angle of triangle is equal to the sum of the two opposite internal angles.

Thus in the case of the triangle shown alongside it says that

$$\angle SRP = \angle RPQ + \angle RQP,$$



that is,

$$t = u + v.$$

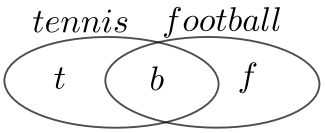
Show how the External Angle Theorem may be deduced from the theorem that says that the sum of the angles in a triangle is 180°.

15. A group of 42 children all play tennis or football, or both sports. The same number play tennis as play just football. Twice as many play both tennis and football as play just tennis.
How many of the children play football?

A 7 B 14 C 21 D 28 E 35

SOLUTION **E**

We let t be the number of children that only play tennis, f be the number of children that only play football, and b be the number of children that play both tennis and football.



It follows that the total number of children who play football is $b + f$.

There are 42 children in the group. Therefore

$$t + b + f = 42. \qquad (1)$$

The number who play tennis is the same as the number who just play football. Therefore

$$t + b = f. \qquad (2)$$

Twice as many play both tennis and football as play just tennis. Therefore

$$b = 2t. \qquad (3)$$

Substituting from (2) in (1), we obtain

$$f + f = 42,$$

from which it follows that

$$f = 21. \qquad (4)$$

Substituting from (3) and (4) in (2), we obtain

$$t + 2t = 21,$$

That is,

$$3t = 21.$$

It follows that

$$t = 7.$$

Hence, by (3),

$$b = 14. \qquad (5)$$

Therefore, by (4) and (5),

$$\begin{aligned} b + f &= 14 + 21 \\ &= 35. \end{aligned}$$

Therefore the number of children who play football is 35.

FOR INVESTIGATION

15.1 How many children play tennis?

16. You are given the sequence of digits "0625", and can insert a decimal point at the beginning, at the end, or at any of the other three positions.
Which of these numbers can you *not* make?

A $\frac{6}{25}$ B $\frac{5}{8}$ C $\frac{1}{16}$ D $\frac{25}{4}$ E 25^2

SOLUTION **A**

We have

$$.0625 = \frac{625}{10\,000} = \frac{25 \times 25}{25 \times 400} = \frac{25}{400} = \frac{25}{25 \times 16} = \frac{1}{16}.$$

Similarly, we have

$$0.625 = \frac{625}{1000} = \frac{5}{8},$$
$$06.25 = \frac{625}{100} = \frac{25}{4},$$
$$062.5 = \frac{625}{10} = \frac{125}{2},$$

and

$$0625 = 25^2.$$

We therefore see that of the given options it is A that cannot be made.

Note: Of course, we normally write 0.0625, 6.25, 62.5 and 625, respectively, instead of .0625, 06.25, 062.5 and 0625.

FOR INVESTIGATION

16.1 Given the sequence of digits "1275", which numbers can be made by inserting a decimal point at the beginning, at the end, or at any of the other three positions?

16.2 Write each of the following decimal numbers as fractions in the form $\frac{a}{b}$, where a and b are positive integers with no common factor other than 1.

(a) 1.25,

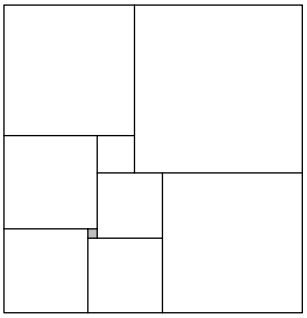
(b) 0.125,

(c) 0.0125,

(d) 0.00125.

17. In 1925, Zbigniew Morón published a rectangle that could be dissected into nine different sized squares as shown in the diagram. The lengths of the sides of these squares are 1, 4, 7, 8, 9, 10, 14, 15 and 18.
What is the area of Morón's rectangle?

A 144 B 225 C 900 D 1024
E 1056



SOLUTION **E**

METHOD 1

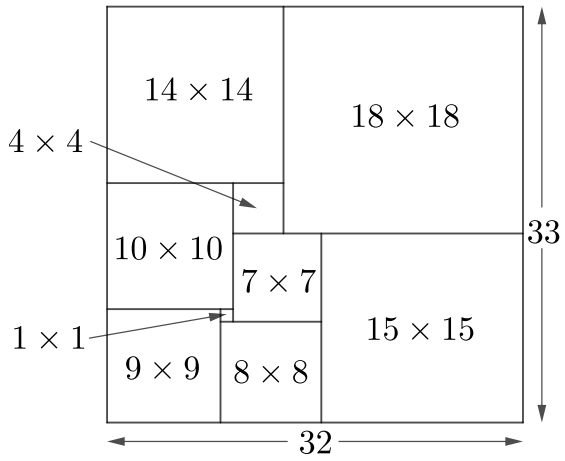
The area of the rectangle is the sum of the areas of the squares that make it up. These squares have side lengths 1, 4, 7, 8, 9, 10, 14, 15 and 18. Hence the sum of their areas is given by

$$\begin{aligned} &1^2 + 4^2 + 7^2 + 8^2 + 9^2 + 10^2 + 14^2 + 15^2 + 18^2 \\ &= 1 + 16 + 49 + 64 + 81 + 100 + 196 + 225 + 324 \\ &= 1056. \end{aligned}$$

METHOD 2

In this method we avoid most of the arithmetic by finding how the squares fit inside the rectangle. From this we can deduce the dimensions of the rectangle.

It can be checked that the sizes of the squares shown in the diagram of the question are as shown in the diagram below.



From this diagram we see that the rectangle has width 32 and height 33. Therefore the area of the rectangle is given by

$$32 \times 33 = 1056.$$

NOTE

Zbigniew Morón's rectangle, which is divided into squares of different sizes, has dimensions 32×33. So it is nearly square. It raised the question of whether a square could be divided into squares of different sizes. Such a square is called a *perfect squared square*.

R. L. Brooks, C. A. B. Smith, A. H. Stone and W. T. Tutte at Cambridge University between 1936 and 1938 considered this problem. They converted the problem into one involving electrical circuits. Using Kirchhoff's laws for electrical circuits, they found a perfect squared square. Their example divided a square into 69 smaller squares of different sizes.

In 1978, A. J. W. Duijvestijn discovered a perfect squared square with the least possible number of smaller squares. Using a computer search he found a that a square with side length 112 may be divided into 21 smaller squares with different integer side lengths. He also proved that 21 squares is the smallest possible number of squares for such a construction.

However 112 is not the smallest possible side length, as Duijvestijn also found two perfect squared squares of sides 110 but each used 22 squares. In 1999, I. Gambini proved 110 is the smallest possible side length for a perfect squared square in which all the squares have integer side lengths.

18. How many two-digit primes have both their digits non-prime?

A 6 B 5 C 4 D 3 E 2

SOLUTION **B**

The digits that are not primes are 0, 1, 4, 6, 8 and 9. However, the digit 0 cannot be the tens digit of a two-digit number.

If the units digit (sometimes called the ones digit) of a two-digit integer is even, the number itself is even and hence is not prime.

Hence we need only consider those two-digit numbers whose units digits are 1 or 9, and whose tens digits are 1, 4, 6, 8 or 9.

Therefore the numbers we need to consider are 11, 19, 41, 49, 61, 69, 81, 89, 91 and 99.

It can be seen that, of these, 69, 81 and 99 are not prime because they are multiples of 3, and 49 and 91 are not prime because they are multiples of 7.

This leaves 11, 19, 41, 61 and 89 as the only two-digit primes whose digits are both non-primes.

Therefore there are 5 such numbers.

FOR INVESTIGATION

18.1 Express each of 49, 69, 81, 91 and 99 as products of primes.

18.2 Check that 11, 19, 41, 61 and 89 are all primes.

18.3 Find all the two-digit primes each of whose digits is a prime.

18.4 Find all the three-digit primes each of whose digits is not a prime.

18.5 Find all the three-digit primes each of whose digits is a prime.

19. In the table shown, the sum of each row is shown to the right of the row and the sum of each column is shown below the column.
What is the value of L?

J	K	J	5
K	K	L	13
L	J	L	15
11	7	15	

A 1 B 2 C 3 D 5 E 7

SOLUTION **E**

Note: Each of the row and column totals gives us an equation. The totals for row 3 and column 3 lead to the same equation $J + 2L = 15$, but this still means that we have five equations relating the three unknowns J, K and L.

In turns out that there are nine ways to use three of these equations to deduce the values of J, K and L. Thus there are many alternatives to the solution given here.

From the second row we have

$$2K + L = 13 \qquad (1)$$

and from the second column

$$2K + J = 7. \qquad (2)$$

By subtracting equation (2) from equation (1) we obtain

$$L - J = 6. \qquad (3)$$

From the third row, we have

$$2L + J = 15. \qquad (4)$$

Adding equations (3) and (4) gives

$$3L = 21,$$

from which it follows that

$$L = 7.$$

FOR INVESTIGATION

19.1 Find the values of J and K.

19.2 Show that from the three equations $J + K + L = 11$, $J + 2K = 7$ and $J + 2L = 15$ corresponding to the column totals, it is *not* possible to deduce the values of J, K and L.

19.3 In the table shown on the right, the sum of each row is shown to the right of the row, and the sum of each column is shown below the column.

What is the value of $U + K + M + T$?

U	K	M	M	16
U	K	K	T	19
U	U	M	T	22
K	M	T	T	23
18	15	19	28	

20. Edmund makes a cube using eight small cubes. Samuel uses cubes of the same size as the small cubes to make a cuboid twice as long, three times as wide and four times as high as Edmund's cube.
How many more cubes does Samuel use than Edmund?

A 9 B 24 C 64 D 184 E 190

SOLUTION **D**

Edmund's cube which uses 8 small cubes will be a $2 \times 2 \times 2$ cube. That is, it will be 2 cubes long, 2 cubes wide and 2 cubes high.

The cuboid that Samuel makes will therefore be 4 cubes long, 6 cubes wide and 8 cubes high.

Hence Samuel's cuboid will use $4 \times 6 \times 8$ cubes, that is, 192 cubes.

Therefore the number of extra cubes that Samuel uses is $192 - 8 = 184$.

21. The digits of both the two-digit numbers in the first calculation below have been reversed to give the two-digit numbers in the second calculation. The answers to the two calculations are the same.

$62 \times 13 = 806 \qquad 26 \times 31 = 806$

For which one of the calculations below is the same thing true?

A 25×36 B 34×42 C 54×56 D 42×48 E 32×43

SOLUTION **D**

COMMENTARY

One way to answer this question would be to calculate all the products, and then repeat the calculations with the digits reversed.

This method potentially involves a lot of work. If option E is correct, you could find yourself doing 10 long multiplications and without a calculator!

This is a clue that you should look for something better.

The idea we use is just to check the units digit (sometimes called the ones digit) of the products. If two products have different units digits, they must be different.

The units digit of a product is the same as that of the product of the units digits of the numbers that are multiplied.

For example the units digit of 25×36 is the same as the units digit of 5×6, namely 0. Similarly, the units digit of 52×63 is the same as the units digit of 2×3, namely 6.

> Since these two products have different units digits, they cannot be equal. We can tell this without evaluating the two products.

In each case we check the units digits (sometimes called the ones digits) of the two products.

In option A, we have $5 \times 6 = 30$ and $2 \times 3 = 6$. Because 30 and 6 have different units digits, we deduce that $25 \times 36 \neq 52 \times 63$.

In option B, we have $4 \times 2 = 8$ and $3 \times 4 = 12$. We deduce that $34 \times 42 \neq 43 \times 24$.

In option C, we have $4 \times 6 = 24$ and $5 \times 5 = 25$. We deduce that $54 \times 56 \neq 45 \times 65$.

In option D, we have $2 \times 8 = 16$ and $4 \times 4 = 16$. So the units digit of both 42×48 and 24×84 is 6. It is therefore possible that both products have the same value. [Note that you are asked in Problem 21.3 to give an example to show that this does not prove that they are equal.]

In option E, we have $2 \times 3 = 6$ and $3 \times 4 = 12$. We deduce that $32 \times 43 \neq 23 \times 34$.

In the context of the JMC, where you can assume that one of the options is correct, you can stop here. From the calculations above we can conclude that the correct option is D.

Note: You are asked to check this in Problem 21.1 below.

FOR INVESTIGATION

21.1 (a) Check that $42 \times 48 = 24 \times 84$, by evaluating both products.

(b) By factorizing 42, 48, 24 and 84, show that $42 \times 48 = 24 \times 84$ without calculating the answers to these multiplication sums.

21.2 We use the notation 'tu' for the two-digit number with tens digit t and units digit u. Thus 'tu' represents the number $10t + u$.

Show that the test for whether 'ab' × 'cd' is equal to 'ba' × 'dc' is that $ac = bd$.

[Note that in the equation $ac = bd$ by ac is meant the product of the digits a and c and not the two-digit number 'ac'. Likewise, bd means the product of the digits b and d.]

21.3 Give an example of a pair of two-digit numbers 'ab' and 'cd' such that the products 'ab' × 'cd' and 'ba' × 'dc' have the same units digits, but these products are not equal.

22. Harriet has a square piece of paper. She folds it in half to form a rectangle and then in half again to form a second rectangle (which is not a square). The perimeter of the second rectangle is 30 cm.
What is the area of the original square?

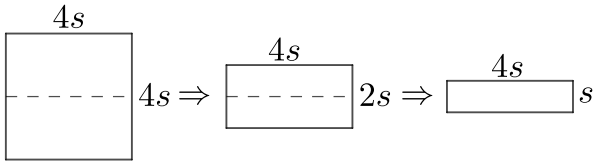
A $36\,\text{cm}^2$ B $64\,\text{cm}^2$ C $81\,\text{cm}^2$ D $100\,\text{cm}^2$ E $144\,\text{cm}^2$

SOLUTION **E**

To avoid fractions, we let the sidelength of the square piece of paper be $4s$ cm.

Harriet folds this piece of paper in half. This results in a $4s$ cm $\times$ $2s$ cm rectangle.

Harriet folds the rectangle in half again to form a second rectangle that is not a square. This results in a $4s$ cm $\times$ s cm rectangle.



Since the perimeter of the second rectangle is 30 cm,

$$4s + s + 4s + s = 30.$$

It follows that

$$10s = 30$$

and hence

$$s = 3.$$

Hence $4s = 12$.

Therefore the original square has side length 12 cm. Hence this square has area $(12 \times 12)\,\text{cm}^2$, that is, $144\,\text{cm}^2$.

23. There is more than one integer, greater than 1, which leaves a remainder of 1 when divided by each of the four smallest primes.
What is the difference between the two smallest such integers?

A 211 B 210 C 31 D 30 E 7

SOLUTION **B**

The four smallest primes are 2, 3, 5 and 7.

An integer n leaves a remainder of 1 when divided by each of these primes provided that $n - 1$ is a multiple of each of them.

So we need to find integers that are multiples of 2, 3, 5 and 7. These are precisely the integers that are multiples of their product $2 \times 3 \times 5 \times 7$.

Now $2 \times 3 \times 5 \times 7 = 210$. So the multiples of 2, 3, 5 and 7 are precisely the multiples of 210.

Therefore the two smallest positive integers that are multiples of 2, 3, 5 and 7 are 210 and 420.

It follows that the two smallest positive integers, greater than 1, that leave a remainder of 1 when divided by each of the four smallest primes are 211 and 421.

The difference between these numbers is $421 - 211$, that is, 210.

FOR INVESTIGATION

23.1 Find the smallest positive integer, greater than 1, that leaves a remainder of 1 when divided by each of the five smallest primes.

NOTE

The idea used in this question and Exercise 23.1 may be used to prove that there are infinitely many different primes, as follows.

Suppose that $\{p_1, p_2, \ldots, p_n\}$ is a finite set of primes. We let N be one more than the product of these primes. That is, we put

$$N = p_1 \times p_2 \times \cdots \times p_n + 1.$$

We see that N leaves remainder 1 when divided by each of the primes $p_1, p_2, \ldots, p_n$. Therefore N is not divisible by any of these primes.

It follows that either N is a prime or it is divisible by a prime that is different from each of $p_1, p_2, \ldots, p_n$.

In either case it follows that there is a prime that is not in the set $\{p_1, p_2, \ldots, p_n\}$.

Hence no finite set of primes contains all the primes. We deduce that the set of primes is infinite.

23.2 Find a finite set of primes, $\{p_1, p_2, \ldots, p_n\}$, such that the number N that is given by $N = p_1 \times p_2 \times \cdots \times p_n + 1$ is *not* prime.

24. Susan is attending a talk at her son's school. There are 8 rows of 10 chairs where 54 parents are sitting. Susan notices that every parent is either sitting on their own or next to just one other person.
What is the largest possible number of adjacent empty chairs in a single row at that talk?

A 3 B 4 C 5 D 7 E 8

SOLUTION **B**

To have the largest number of empty chairs in a single row there would need to be the maximum possible number of parents sitting in the other seven rows.

Because no parent is sitting next to two other parents, there cannot be three consecutive chairs in which parents are sitting.

This means that there can be at most seven chairs that are occupied in each row of ten chairs. An example of how seven chairs in one row might be filled is shown in the diagram alongside.

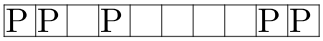
Therefore the maximum number of parents that could be seated in seven rows is $7 \times 7 = 49$.

Since there are 54 seated parents the smallest number of parents that there could be in the remaining row is $54 - 49 = 5$.

In seating five parents in one row there will be five empty chairs. The five parents must be seated in at least three groups.

So there will be at least two gaps in the row and five empty chairs making up these gaps.

Therefore the maximum gap that there can be is four chairs when there are two gaps, with one empty chair in one gap, and four adjacent empty chairs in the other gap. An arrangement of this kind is shown alongside.



Therefore the largest possible number of adjacent empty chairs in a single row is four.

FOR INVESTIGATION

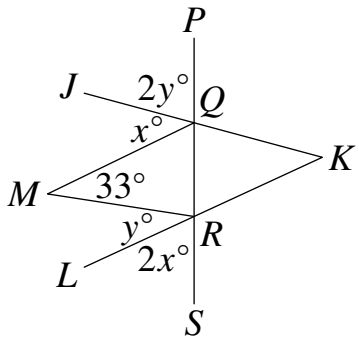
24.1 Suppose that there were only 53 parents who were sitting to hear the talk. What is the largest possible number of adjacent empty chairs in a single row in this case?

24.2 Suppose that there were 55 parents who were sitting to hear the talk. What is the largest possible number of adjacent empty chairs in a single row in this case?

25. In the diagram, $PQRS$, JQK and LRK are straight lines.

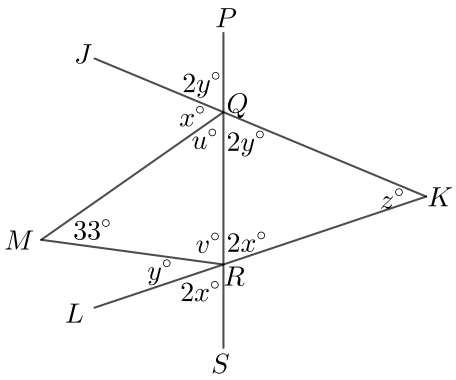
What is the size of the angle JKL?

A 34° B 35° C 36° D 37°
E 38°



SOLUTION **E**

We let $\angle JKL = z°$, $\angle MQR = u°$ and $\angle MRQ = v°$.



Because they are vertically opposite angles, $\angle QRK = \angle LRS = 2x°$.

Similarly, $\angle RQK = \angle JQP = 2y°$.

Because the angles on the line LRK at the point R have sum 180°,

$$y + v + 2x = 180.$$

Similarly,

$$x + u + 2y = 180.$$

Adding these two equations, we obtain

$$3x + 3y + u + v = 360. \qquad (1)$$

Because the sum of the angles in the triangle MRQ is 180°,

$$u + v = 180 - 33 = 147. \qquad (2)$$

Substituting from equation (2) in equation (1) gives

$$3x + 3y + 147 = 360$$

from which it follows that

$$3x + 3y = 360 - 147 = 213.$$

Therefore, dividing both sides of the last equation by 3,

$$x + y = 71. \qquad (3)$$

From the triangle QRK,

$$2x + 2y + z = 180$$

and therefore

$$z = 180 - 2(x + y). \qquad (4)$$

Substituting from (3) in equation (4), we deduce that

$$z = 180 - 2(71) = 180 - 142 = 38.$$

Therefore $\angle JKL = 38°$.