1. Let n be a positive integer and let A be a set of $n+1$ positive integers each of which is at most $2n$.

(a) Show that A contains a pair of integers that differ by n.
(b) Show that A contains a pair of integers that are *coprime* (have highest common factor 1).
(c) Show that A contains a pair of integers one of which divides the other.

2. Suppose that we are given n lines in the plane, such that no two are parallel and no three meet at a point. Let T be the number of distinct regions (of possibly infinite area) which they enclose. What are the possible values for T (in terms of n)?

3. (a) Prove that for all positive integers a, b, c:

$$a+b+c \quad \text{divides} \quad a^3+b^3+c^3-3abc$$

(b) Find all triples of positive integers (x, y, z) such that:

$$x+y+z=10 \quad \text{and} \quad (x-1)^3+(y-2)^3+(z-7)^3=36$$

4. Find all positive integers n such that, for all odd integers a with $a^2 < n$, a divides n.

5. *This question asks you to prove the alternate segment theorem and tangent-secant theorem.*

(a) Let ABC be a triangle. Let Γ be the circle passing through points A, B, C (the *circumcircle* of $\triangle ABC$). Point D is on the tangent to Γ at C such that D and A lie on opposite sides of line BC.

Prove that $\angle BCD = \angle BAC$.

(b) Continuing from the above, assume that $CA \neq CB$, and let CD intersect AB at a point P. Show that $PA \cdot PB = PC^2$.

6. Let x be a real number such that $t = x + x^{-1}$ is an integer greater than 2.

(a) Prove that $t_n = x^n + x^{-n}$ is an integer for all positive integers n.

(b) Determine the values of n for which t divides t_n.

[*BMO1 2014/15 Q4*]

7. Find all positive integers n such that both $n+2008$ divides n^2+2008 and $n+2009$ divides n^2+2009.

[*BMO1 2008/09 Q4*]

8. A positive integer is called charming if it is equal to 2 or is of the form $3^i \cdot 5^j$ where i and j are non-negative integers.

(a) Show that for every positive integer m we can find a charming integer C such that $m < C \leqslant 2m$.

(b) Prove that every positive integer can be written as a sum of different charming integers.

[*BMO1 2015/16 Q6*]