Andrew Jobbings Senior Kangaroo

a member of the Association Kangourou sans Frontières

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Solutions

These are polished solutions and do not illustrate the process of failed ideas and rough work by which candidates may arrive at their own solutions.

It is not intended that these solutions should be thought of as the 'best' possible solutions and the ideas of readers may be equally meritorious.

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1. The number x satisfies $x^2 - 4x + 2 = 0$. What is the value of $x + \frac{2}{x}$?

SOLUTION **004**

As $x \neq 0$ we may divide $x^2 - 4x + 2 = 0$ by x to give $x - 4 + \frac{2}{x} = 0$, and then $x + \frac{2}{x} = 4$.

2. The sum of the ages of Tom and John is 23, the sum of the ages of John and Alex is 24 and the sum of the ages of Tom and Alex is 25. What is the sum of the squares of their three ages?

SOLUTION **434**

Let the ages of Tom, John and Alex be T, J and A respectively. We have $T + J = 23$ and $T + A = 25$. Subtracting these equations we have $A - J = 2$. Solving this, and $J + A = 24$ simultaneously, we have $J = 11$, $A = 13$ and hence $T = 12$. Therefore, the sums of the squares of their ages is $11^2 + 12^2 + 13^2 = 434$.

3. Zia conducted a study of the number of coins each member of her club had in their pocket. To her surprise, nobody had zero coins and exactly seven people had only one coin each. Everybody else had at least two coins. She drew a pie chart of her results, and to her surprise found each sector had an angle which was an integer ten degrees larger than its immediate predecessor. The smallest sector in the pie chart had an angle of 35^o. How many people are in Zia's club?

SOLUTION **72**

The sector angles, measured in degrees, are 35, 45, 55, 65, 75, and 85, which sum to 360. Since the smallest sector represents a frequency of seven, we know that five degrees corresponds to one member of the club.
Therefore, the number of people in Zia's club is $360 \div 5 = 72$.

4. The rectangles S_1 and S_2 are drawn on a set of coordinate axes. S_1 has vertices at $(0, y)$; (x, y); $(x, 4)$; and $(0, 4)$, where $x < 9$ and $y < 4$. S_2 has vertices at $(x, 0)$; $(9, 0)$; $(9, y)$; and (x, y). The rectangles are equal in area. What is the value of $\frac{360x}{y}$?

SOLUTION **810**

Rectangles S_1 and S_2 have areas $x(4 - y)$ and $y(9 - x)$ respectively.
Equating these, $4x - xy = 9y - xy$.
Therefore, $4x = 9y$, which yields $360x = 810y$ giving the result of $\frac{360x}{y} = 810$.

5. Paul is painting a wall. He knows the area of the wall is 1920 square metres, correct to the nearest ten. He uses tins of paint, each of which can cover 18 square metres, correct to the nearest integer.
He needs to paint the wall completely, and still have at least half a tin of paint left over for any minor repairs.
What is the smallest number of tins he must buy to be sure of having enough paint?

SOLUTION **111**

The number of tins required to certainly cover the wall will be the maximum possible size of the wall divided by the minimum possible amount of paint per tin, i.e. $\frac{1925}{17.5} = \frac{19250}{175} = \frac{770}{7} = 110$ (exactly). Because the calculation uses an exact number of tins, and as he requires a surplus of half a tin for any touch-ups he should buy one additional tin. Therefore, he should buy 111 tins of paint.

6. A cube is dissected into 6 pyramids by connecting a given point in the interior of the cube with each vertex of the cube, so that each face of the cube forms the base of a pyramid. The volumes of five of these pyramids are 200, 500, 1000, 1100 and 1400. What is the volume of the sixth pyramid?

SOLUTION **600**

Each of the cube's faces forms the base of one of the six pyramids. Any two pyramids formed from opposite faces will have a combined volume of $\frac{1}{3}$ × area of cube face × distance between opposite faces $= \frac{1}{3} \times s^3$, where s is the cube's side-length.
Pairing the known pyramid volumes we see that $200 + 1400 = 500 + 1100 = 1600$. The missing pyramid therefore has volume $1600 - 1000 = 600$.

7. The positive integer N has exactly six distinct (positive) factors including 1 and N. The product of five of these is 6075. What is the value of the sixth factor?

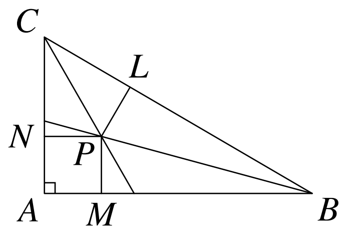
SOLUTION **015**

We determine that $6075 = 3^5 \times 5^2$. This means that 1, 3, 5 and 15 must be factors of N.
Since 3^5 is a factor of 6075, 9 is also a factor of N and so is $9 \times 5 = 45$.
$1 \times 3 \times 5 \times 9 \times 15 \times 45 = 3^6 \times 5^3$. Therefore, the missing factor is $3^6 \times 5^3 \div \left(3^5 \times 5^2\right) = 15$.

8. In a right-angled triangle ABC (with right angle at A) the bisectors of the acute angles intersect at point P. The distance from P to the hypotenuse is $\sqrt{80000}$. What is the distance from P to A?

Solution **400**

Drop the perpendiculars from P to AB, meeting at M; and from P to AC, meeting at N; and from P to BC, meeting at L. Triangles BLP and BMP are congruent, since both have a right angle and one-half of the bisected angle at A, and share edge AP.



Therefore PM is $\sqrt{80000}$ in length. A similar argument is used to prove PN is also $\sqrt{80000}$. The distance AP is then found by Pythagoras' equation in triangle APM: $AP = \sqrt{80000 + 80000} = \sqrt{160000} = 400$.

9. Using the 24-hour clock, the time 16:11:22 on the date 16/11/22 (i.e. the 16th of November 2022) has hours, minutes and seconds equal to date, month and (two-digit) year respectively. Let S be the number of seconds which will elapse after that date and time until this phenomenon next occurs.
What is the value of $\sqrt{S}$?

Solution **300**

The next such time and date will be 17:11:22 on 17/11/22. This is one day and one hour later.
The number of seconds which will elapse is $25 \times 60 \times 60 = 90000$.
Therefore, $\sqrt{S} = 5 \times 60 = 300$.

10. The equations $x^2 + ax + b = 0$ and $x^2 + bx + a = 0$ both have real roots. It is known that the sum of the squares of the roots of the first equation is equal to the sum of the squares of the roots of the second one, and that $a \neq b$. Find the value of $500 + 10(a + b)$.

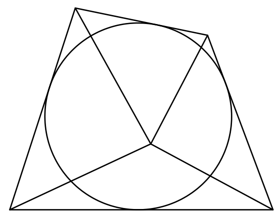
Solution **480**

Let p, q be the roots of the first equation, and r, s the roots of the second.
Then $(x - p)(x - q) = x^2 + ax + b$. After expanding the brackets and comparing coefficients we have $a = -p - q$ and $b = pq$.
A similar analysis of the second equation yields $b = -r - s$ and $a = rs$.
Now, $(-p - q)^2 = p^2 + q^2 + 2pq = p^2 + q^2 + 2b$.
Similarly, $(-r - s)^2 = r^2 + s^2 + 2rs = r^2 + s^2 + 2a$.
Therefore, $(-p - q)^2 - 2b = (-r - s)^2 - 2a$ (since $p^2 + q^2 = r^2 + s^2$).
Therefore, $a^2 - 2b = b^2 - 2a$ and hence $a^2 - b^2 = 2b - 2a = 2(b - a)$.
Therefore $(a - b)(a + b) = -2(a - b)$ and hence $a + b = -2$, since we know that $a - b \neq 0$.
We conclude that $500 + 10(a + b) = 500 - 20 = 480$.

11. A quadrilateral contains an inscribed circle (i.e. a circle tangent to the four sides of the quadrilateral). The ratio of the perimeter of the quadrilateral to that of the circle is $4 : 3$. The ratio of the area of the quadrilateral to that of the circle, expressed in its lowest terms, is $a : b$. What is the value of $100a + 49b$?

Solution **547**

Let the radius of the circle be 3. Then its circumference is 6π and the quadrilateral's perimeter is 8π. The circle's area is 9π. We can determine the quadrilateral's area by dividing it into four triangles, as shown in the example. Each triangle has area $\frac{1}{2} \times 3 \times s$, where s is the side shared by the triangle and the quadrilateral.



By summing the triangles' areas we find the quadrilateral's area, as $\frac{1}{2} \times 3\times$ (the quadrilateral's perimeter), this is $\frac{1}{2} \times 3 \times 8\pi = 12\pi$. The required ratio of areas is $12\pi : 9\pi = 4 : 3$. Therefore $a = 4$, $b = 3$, and $100a + 49b = 547$.

12. Five cards have the numbers 101, 102, 103, 104 and 105 on their fronts.

101	102	103	104	105

On the reverse, each card has one of five different positive integers: a, b, c, d and e respectively. We know that $a + 2 = b - 2 = 2c = \frac{d}{2} = e^2$.
Gina picks up the card which has the largest integer on its reverse. What number is on the front of Gina's card?

Solution **104**

We know that $d = 4c$, $d = 2a + 4$, and $d = 2e^2$.
So we can quickly see that $d > c$, $d > a$ and $d > e$.
Also, $d = 2b - 4 > b$ for $b > 4$. We can eliminate the $b \leq 4$ case, for b cannot be odd (for then c would not be an integer); b cannot be 4 (for then a would be zero); and b cannot be 2 (for then c would be zero). Therefore, $d > b$.
Therefore d is the largest of a, b, c, d and e, so Gina has 104 on the front of her card.

13. A sequence is given by $x_1 = 2$, $x_{n+1} = \frac{{x_n}^{x_n}}{32^{n-1}}$. What is $\frac{x_4}{128}$?

Solution **128**

We determine each term in sequence.
$x_2 = \frac{2^2}{32^0} = \frac{4}{1} = 4$; $x_3 = \frac{4^4}{32^1} = \frac{2^8}{2^5} = 2^3 = 8$; $x_4 = \frac{8^8}{32^2} = \frac{2^{24}}{2^{10}} = 2^{14}$.

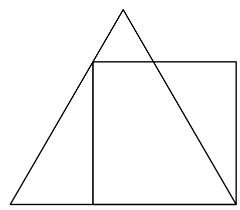
Therefore, $\frac{x_4}{128} = 2^{14} \div 2^7 = 2^7 = 128$.

14. A class consists of pupils who each wear either a red tie or a green tie. There are two more pupils wearing green ties than wearing red ties. Two pupils are chosen at random. The probability of selecting two pupils of different tie-colour is exactly treble the probability of selecting two pupils who both wear red ties. Given that R is the number of pupils wearing a red tie, and that $R > 0$, determine $R^3 + R^2 + R$.

Solution **399**

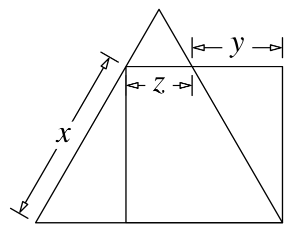
The number of pupils wearing a green tie is $R + 2$, and the total number of pupils in the class is $R + R + 2 = 2R + 2$. Let the probability of selecting two red-tie wearing pupils be q.
Then $\frac{R}{2R+2} \times \frac{R-1}{2R+1} = q$.
Also, $\frac{R}{2R+2} \times \frac{R+2}{2R+1} + \frac{R+2}{2R+2} \times \frac{R}{2R+1} = 3q$.
Eliminating q we have $3\left(\frac{R}{2R+2} \times \frac{R-1}{2R+1}\right) = \frac{R}{2R+2} \times \frac{R+2}{2R+1} + \frac{R+2}{2R+2} \times \frac{R}{2R+1}$.
Therefore, $3R(R - 1) = 2R(R + 2)$ and hence $3R^2 - 3R = 2R^2 + 4R$.
This quadratic equation simplifies to $R^2 - 7R = 0$ and has solutions $R = 0$ and $R = 7$.
We are given that $R > 0$, so $R = 7$ and $R^3 + R^2 + R = 343 + 49 + 7 = 399$.

15. The perimeter of the square in the figure is 40. The perimeter of the larger equilateral triangle in the figure is $a + b\sqrt{p}$, where p is a prime number. What is the value of $7a + 5b + 3p$?



Solution **269**

Define lengths x, y and z as shown in the diagram, so that the larger equilateral triangle has side length $x + z$.



We have $x = \frac{10}{\sin 60} = \frac{20}{\sqrt{3}}$.
Also, $y = \frac{10}{\tan 60} = \frac{10}{\sqrt{3}}$ and hence $z = 10 - \frac{10}{\sqrt{3}}$.
The perimeter of the larger equilateral triangle is

$$3(x + z) = 3\left(\frac{20}{\sqrt{3}} + 10 - \frac{10}{\sqrt{3}}\right) = 3\left(\frac{10}{\sqrt{3}} + 10\right) = 30 + 10\sqrt{3}.$$

Therefore, $a = 30$, $b = 10$ and $c = 3$ and so $7a + 5b + 3p = 210 + 50 + 9 = 269$.

16. On the island of Friends and Foes, every citizen is either a Friend (who always tells the truth) or a Foe (who always lies). Seven citizens are sitting in a circle. Each declares "I am sitting between two Foes". How many Friends are there in the circle?

Solution **003**

If every citizen is a foe then all are telling the truth, which is a contradiction. Therefore, there is at least one friend. Suppose the citizens are labelled ABCDEFG and sat in the same order, with A (a friend) sat next to B and G. Then B and G are both foes. We now consider C and F.
If C is a foe, then D must be a friend (else C would speak the truth). This makes E a foe, and F must be a friend since F is sitting between two foes, so speaks the truth. In this scenario there are three friends, A, D and F.
If F is a foe, then E must be a friend (else F would speak the truth). This makes D a foe, and C must be a friend since C is sitting between two foes, so speaks the truth. In this scenario there are three friends, A, C and E.
If C and F are both friends then D and E must both be foes. In this scenario there are three friends, A, C and F.
Therefore there are three friends.

17. What is the smallest three-digit positive integer which can be written in the form pq^2r, where p, q and r are distinct primes?

Solution **126**

We consider multiples of q^2, for increasing values of q.
For $q = 2$, we consider numbers of the form $4k$, where k is odd (since neither p nor r can then be 2).
$100 = 2^2 \times 5^2$; $108 = 2^2 \times 3^3$; $116 = 2^2 \times 29$; $124 = 2^2 \times 31$; $132 = 2^2 \times 3 \times 11$, so 132 is the first multiple of 2^2 of the required form.
For $q = 3$ we consider multiples of 9.
$108 = 2^2 \times 3^3$; $117 = 3^2 \times 13$; $126 = 2 \times 3^2 \times 7$, so 126 is the first multiple of 3^2 of the required form.
For $q = 5$ we consider multiples of 25.
$100 = 2^2 \times 5^2$; $125 = 5^3$; and subsequent multiples are larger than 126.
For $q = 7$ we consider multiples of 49. However, the smallest three-digit multiple, 147, is already larger than 126.
For $q = 11$ we consider multiples of 121. However, $121 = 11^2$ and subsequent multiples are larger than 126.
For all other possible values of q, q^2 is larger than 126. Therefore, 126 is the smallest such number.

18. The number $\dfrac{20! \times 22!}{16! \times 11!}$ has N prime factors, which are not necessarily distinct.

What is the value of $N(N-2)$?

Solution **960**

We can rewrite the fraction as $\frac{20!}{11!} \times \frac{22!}{16!}$.
By writing each factorial out as a product, and cancelling like factors top and bottom of each fraction, this simplifies to
$20 \times 19 \times 18 \times 17 \times 16 \times 15 \times 14 \times 13 \times 12 \times 22 \times 21 \times 20 \times 19 \times 18 \times 17$.
We can count how many non-distinct prime factors each number in the above product has, and find the sum.
Taking the numbers in turn, $N = 3 + 1 + 3 + 1 + 4 + 2 + 2 + 1 + 3 + 2 + 2 + 3 + 1 + 3 + 1 = 32$.
Therefore, $N(N-2) = 32 \times 30 = 960$.

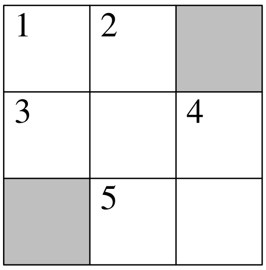
19. How many different real number solutions are there to the following equation?

$$\left(x^2 - 8x + 15\right)^{\left(x^5 - 5x^3 + 4x\right)} = 1$$

Solution **008**

Some solutions occur when $x^5 - 5x^3 + 4x = 0$, provided $x^2 - 8x + 15 \neq 0$ simultaneously. Therefore we consider $x(x^4 - 5x^2 + 4) = 0$, i.e. $x(x^2-1)(x^2-4) = 0$ i.e. $x(x-1)(x+1)(x-2)(x+2) = 0$.
The solutions to this are -2, -1, 0, 1 and 2, none of which makes $x^2 - 8x + 15 = 0$, so all five of these are solutions.
Some solutions occur when $x^2 - 8x + 15 = 1$. This has two distinct solutions $x = 4 \pm \sqrt{2}$.
Some solutions may occur when $x^2 - 8x + 15 = -1$. This is equivalent to $x^2 - 8x + 16 = 0$ which has exactly one solution, namely $x = 4$. This will only solve the original equation if the index $x^5 - 5x^3 + 4x$ is even when $x = 4$. At $x = 4$, the index has a value of $4^5 - 5 \times 4^3 + 4 \times 4 = 720$, which is even. Therefore $x = 4$ is a solution.
Therefore, there are $5 + 2 + 1 = 8$ different solutions.

20. Each cell in this cross-number can be filled with a non-zero digit such that all of the conditions in the clues are satisfied. The digits used are not necessarily distinct. Determine T, the sum of all six answers to the clues. The answer to this question is $0.5 \times T$.



ACROSS:
1. Two less than a prime.
3. Exactly 100 more than its largest proper factor.
5. A multiple of 13.

DOWN:
1: A fourth power.
2: A cube.
4: Not prime, not square, not even.

Solution **582**

1 DOWN is either 16 or 81. If 3 ACROSS begins with a six then its largest proper factor is at most 349, which violates the clue. Therefore 3 ACROSS begins with a one, and 1 DOWN must be 81.
1 ACROSS is either 81 or 87. If 2 DOWN begins with a one then it is 125 and 5 ACROSS must be 52. But this would mean 4 DOWN ended with a two, which violates its clue. Therefore 1 ACROSS must be 87, in turn meaning 2 DOWN is 729 and 5 ACROSS is 91.
4 DOWN ends in a one so can only be 21, 51 or 91. This means 3 ACROSS has possible answers of 122, 125 and 129. Of these, only 125 satisfies the clue.
The total number of all the answers in the grid is 87 + 125 + 91 + 81 + 729 + 51 = 1164. The required quantity is 1164 ÷ 2 = 582.

