United Kingdom
Mathematics Trust

Senior Mathematical Challenge

Organised by the United Kingdom Mathematics Trust

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Solutions and investigations

October 3rd, 2023

These solutions augment the shorter solutions also available online. The shorter solutions sometimes leave out details. The solutions given here are full solutions, as explained below. In some cases alternative solutions are given. There are also many additional problems for further investigation. We welcome comments on these solutions and the additional problems. Please send them to challenges@ukmt.org.uk.

The Senior Mathematical Challenge (SMC) is a multiple-choice paper. Sometimes you can find the correct answers by working backwards from the given alternatives, or by showing that four of them are not correct. This can be a sensible thing to do in the context of the SMC.

However, this does not provide a full mathematical explanation that would be acceptable if you were just given the questions without any alternative answers. So we aim at including for each question a complete solution with each step explained (or, occasionally, left as an exercise), and not based on the assumption that one of the given alternatives is correct. We hope that these solutions provide a model for the type of written solution that is expected when presenting a complete solution to a mathematical problem (for example, in the British Mathematical Olympiad, the Mathematical Olympiad for Girls and similar competitions).

Enquiries about the Senior Mathematical Challenge should be sent to:

SMC, challenges@ukmt.org.uk

www.ukmt.org.uk

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
C	C	D	C	B	B	D	A	D	B	B	E	A	E	D	A	E	C	E	C	D	C	D	B	A

1. What is the value of $\sqrt{\dfrac{2023}{2+0+2+3}}$?

A 13 B 15 C 17 D 19 E 21

SOLUTION **C**

$2+0+2+3=7$ and $2023 \div 7 = 289 = 17^2$. Therefore

$$\sqrt{\frac{2023}{2+0+2+3}} = \sqrt{17^2} = 17.$$

FOR INVESTIGATION

1.1 Which is the smallest integer n with $n > 2023$ which has the property that n is divisible by the sum of its digits?

1.2 Note that since $2023 \div 7 = 17^2$, it follows that $2023 = 7 \times 17^2$ and that 7 and 17 are primes.

Which is the smallest integer n with $n > 2023$ which can be expressed as $p \times q^2$, where p and q are different primes?

1.3 Which is the smallest integer n with $n > 2023$ which can be expressed as $p \times q^2$, where p and q are different primes, and p is the sum of the digits of n?

2. What is the difference between one-third and 0.333?

A 0 B $\dfrac{3}{1000}$ C $\dfrac{1}{3000}$ D $\dfrac{3}{10000}$ E $\dfrac{1}{30000}$

SOLUTION **C**

Expressed as fractions, one-third is $\frac{1}{3}$ and 0.333 is $\frac{333}{1000}$. Therefore, the difference between one-third and 0.333 is

$$\frac{1}{3} - \frac{333}{1000} = \frac{1000}{3000} - \frac{999}{3000} = \frac{1}{3000}.$$

FOR INVESTIGATION

2.1 (a) What is the difference between one-third and 0.3333 ?

(b) What is the least positive k such that the difference between one-third and $0.\overbrace{33\ldots33}^{k}$ is less than 10^{-6} ?

2.2 What is the difference between $\dfrac{22}{7}$ and 3.141 ?

2.3 What is the difference between 1 and $0.\dot{9}$?

3. The base of a triangle is increased by 20% and its height is decreased by 15%.

What happens to its area?

A It decreases by 3% B It remains the same C It decreases by 2%
D It increases by 2% E It increases by 5%

Solution **D**

Suppose the original triangle has base b and height h. The area of this triangle is X, where $X = \frac{1}{2}(b \times h)$.

When the base of the triangle is increased by 20%, its base becomes $b' = \frac{6}{5}b$. When its height is decreased by 15%, its height becomes $h' = \frac{17}{20}h$.

Therefore the area of the changed triangle is X', where $X' = \frac{1}{2}(b' \times h') = \frac{1}{2}(\frac{6}{5}b \times \frac{17}{20}h) = (\frac{6}{5} \times \frac{17}{20})(\frac{1}{2}(b \times h)) = \frac{102}{100}X$.

Therefore the effect of the changes is to increase the area of the triangle by 2%.

For investigation

3.1 The base of a triangle is decreased by 15% and its height is increased by 20%. What happens to its area?

3.2 The base of a triangle is increased by 20% and its height is decreased by 20%. What happens to its area?

3.3 The base of a triangle is increased by 20%. By what percentage should its height be decreased to keep the area unchanged?

4. In 2016, the world record for completing a 5000m three-legged race was 19 minutes and 6 seconds. It was set by Damian Thacker and Luke Symonds in Sheffield.

What was their approximate average speed in km/h?

A 10 B 12 C 15 D 18 E 25

Solution **C**

The average speed running 5000 m in 19 minutes and 6 seconds is approximately the same as running this distance in 20 minutes. So their average speed was approximately $\dfrac{5000}{20}$ metres per minute, that is 250 metres per minute.

A speed of 250 metres per minute is the same as 60×250 metres per hour, that is, 15 000 metres per hour. This is the same as 15 km/h.

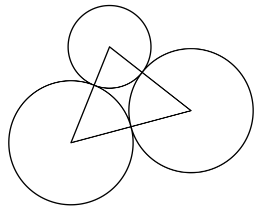
For investigation

4.1 What is the approximation 15 km/h as a percentage of the actual average speed of Damian Thacker and Luke Symonds?

5. Three circles with radii 2, 3 and 3 touch each other, as shown in the diagram.

What is the area of the triangle formed by joining the centres of these circles?

A 10 B 12 C 14 D 16 E 18

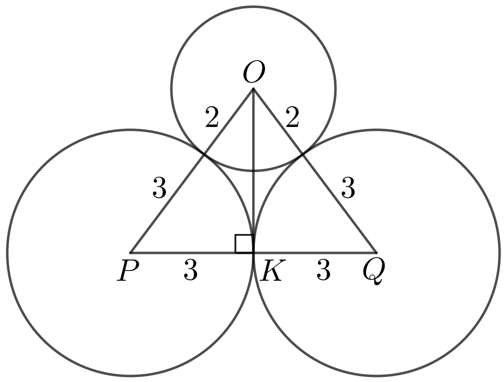


Solution **B**

We let O be the centre of the circle with radius 2, and let P and Q be the centres of the circles with radius 3.

The line joining the centres of touching circles goes through the point where the circles touch. [You are asked to prove this in Problem 5.2.] It follows that both OP and OQ have length $2 + 3 = 5$, and PQ has length $3 + 3 = 6$.

Let K be the midpoint of PQ.



The triangles OPK and OQK are congruent (SSS), and therefore the angles $\angle OKP$ and $\angle OKQ$ are equal, and therefore they are both 90°.

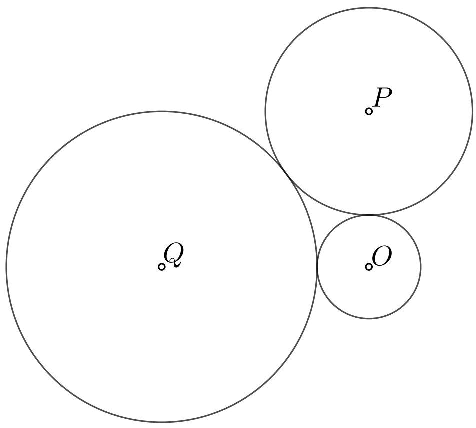
By Pythagoras' Theorem, applied to the triangle OKP, we have $OK^2 = OP^2 - PK^2 = 5^2 - 3^2 = 25 - 9 = 16$. Therefore $OK = 4$.

The triangle OPQ has base PQ of length 6, and height OK of length 4. Therefore the area of this triangle is $\frac{1}{2}(6 \times 4) = 12$.

For investigation

5.1 Three circles with centres O, P and Q with radii 1, 2 and 3, respectively, touch each other as shown.

What is the area of the triangle OPQ ?



5.2 Prove that the line joining the centres of touching circles goes through the point where the circles touch.

6. How many lines of three adjacent cells can be chosen from this grid, horizontally, vertically or diagonally, such that the sum of the numbers in the three cells is a multiple of three?

A 30 B 24 C 18 D 12 E 6

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16

Solution **B**

Suppose first that in a line of three numbers, adjacent numbers have the same difference. Let n be the first of these numbers, and d be the common difference. Then these numbers are n, $n + d$ and $n + 2d$. The sum of these numbers is $3n + 3d$ and therefore is a multiple of 3.

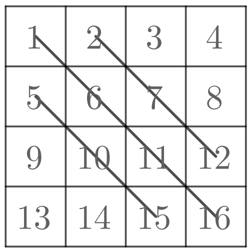
Adjacent numbers in each row have a common difference 1. Therefore, the sum of the numbers in three adjacent cells in the same row is always a multiple of 3.

There are two lines of three adjacent cells in each row, for example 1,2,3 and 2,3,4 in the top row.

Therefore, in the 4 rows there are $4 \times 2 = 8$ lines of three adjacent cells such that the sum of the numbers in these cells is a multiple of 3.

Adjacent numbers in each column differ by 4. Hence, it follows similarly, that there are 8 lines of three adjacent cells in the same column such that the sum of the numbers in these cells is a multiple of 3.

We now consider the diagonals from top left to bottom right. Adjacent numbers in these diagonals each column differ by 5. Therefore the sum of numbers in three adjacent cells in the same diagonal is always a multiple of 3.



One of these diagonals contains 4 numbers. There are 2 lines of three adjacent cells in this diagonal whose sum is a multiple of 3. There are 2 of these diagonals containing three numbers. Therefore, altogether, there are 4 lines of three adjacent numbers on theses diagonals whose sum is a multiple of 3.

Similarly, on the diagonals from top right to bottom left adjacent numbers have a common difference 3, and therefore there are 4 lines of three adjacent numbers on these diagonals whose sum is a multiple of 3.

Hence, in total, there are $8 + 8 + 4 + 4 = 24$ lines of three adjacent cells whose sum is a multiple of 3.

For investigation

6.1 How many lines of three adjacent cells can be chosen from the grid, horizontally, vertically or diagonally, such that the sum of the numbers in the three cells is a multiple of six?

6.2 How many lines of four adjacent cells can be chosen from the grid, horizontally, vertically or diagonally, such that the sum of the numbers in the four cells is a multiple of four?

6.3 In how many ways can three different cells in any position, be chosen from the grid such that sum of the numbers in the three cells is a multiple of three?

7. A sequence begins 2023, 2022, 1, ... After the first two terms, each term is the positive difference between the previous two terms.

What is the value of the 25^{th} term?

A 2010 B 2009 C 2008 D 2007 E 2006

Solution **D**

The sequence begins

$$2023,\ 2022,\ 1,\ 2021,\ 2020,\ 1,\ 2019,\ 2018,\ 1,\ \ldots.$$

From this it seems that, in general, for each non-negative integer k, the terms in positions $3k+1$, $3k+2$ and $3k+3$ are $2023-2k$, $2023-2k-1$ and 1. [In fact, this holds only provided $2023-2k-1 \geq 0$, that is, only for $k \leq 1011$. See Problems 7.1 and 7.4.]

Now $25 = 3\times8+1$. Therefore, by putting $k = 8$, we deduce that the 25 th term is $2023-2\times8 = 2007$.

For investigation

7.1 (a) What are the 3034 th, 3045 th and 3046 th terms of the sequence of this question?

(b) What is the 5000 th term of the sequence of this question?

7.2 A sequence begins

$$2023,\ 2021,\ 2, \ldots$$

After the first two terms, each term is the positive difference between the previous two terms.

What is the 25 th term of the sequence?

7.3 A sequence begins 2023, s, $2023-s$, ... After the first two terms, each term is the positive difference between the previous two terms.

Which is the positive integer s for which the 25 th term of this sequence is 199?

7.4 We let u_n be the n th term of the sequence of this question.

If you have met the method of *Proof by Mathematical Induction,* use this method to prove that for each non-negative integer k,

$$u_{3k+1} = \begin{cases} 2023-2k, & \text{if } k \leq 1011, \\ 1, & \text{otherwise,} \end{cases}$$

$$u_{3k+2} = \begin{cases} 2023-2k-1, & \text{if } k \leq 1011, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$u_{3k+3} = 1.$$

8. What is the value of $99(0.\dot{4}\dot{9} - 0.\dot{4})$?

A 5 B 4 C 3 D 2 E 1

Solution **A**

The standard method for converting a recurrent decimal to a fraction shows that

$$0.\dot{4}\dot{9} = \frac{49}{99} \quad \text{and} \quad 0.\dot{4} = \frac{4}{9}.$$

[Problem 8.1 asks you to check this.]

Therefore

$$\begin{aligned} 99(0.\dot{4}\dot{9} - 0.\dot{4}) &= 99\left(\frac{49}{99} - \frac{4}{9}\right) \\ &= 99\left(\frac{49}{99} - \frac{44}{99}\right) \\ &= 99\left(\frac{5}{99}\right) \\ &= 5. \end{aligned}$$

For investigation

8.1 Show that $0.\dot{4} = \dfrac{4}{9}$ and $0.\dot{4}\dot{9} = \dfrac{44}{99}$.

8.2 Express the recurring decimal $0.\dot{2}3\dot{4}$ as a fraction in its lowest terms.

8.3 Write the solution to the equation

$$x + 0.0\dot{7} = 0.\dot{1}\dot{3}$$

as a recurring decimal.

8.4 Prove that every recurring decimal may be expressed in the form $\dfrac{p}{q}$ where p and q are integers, with $q > 0$.

9.

Across	**Down**
1. A square	1. Twice a fifth power
3. A fourth power	2. A cube

1	*	2
	(shaded)	
3		

When completed correctly, the cross number is filled with four three-digit numbers.

What digit is *?

A 0 B 1 C 2 D 4 E 6

SOLUTION **D**

When you are faced with a crossnumber, the best strategy is to look for clues where it is easy to find a unique solution.

Among the three-digit integers there are more squares and cubes than fourth and fifth powers. So the best strategy is to begin with 1 Down and 3 Across.

The first few numbers that are twice fifth powers are $2 \times 1^5 = 2 \times 1 = 2$, $2\times2^5 = 2\times32 = 64$, $2\times3^5 = 2\times243 = 486$ and $2\times4^5 = 2\times1024 = 2048$. We can deduce from this that 486 which is the only three-digit number in this list is the answer for 1 Down.

1 4	4	2 1
8	(shaded)	2
3 6	2	5

It follows that 3 Across is a three-digit fourth power with 6 as its hundreds digit. We have $3^4 = 81$, $4^4 = 256$, $5^4 = 625$ and $6^4 = 1296$. We deduce that 3 Across is 625.

We now see that 2 Down is three-digit cube with units digit 5. Hence 2 Down is is $5^3 = 125$.

Finally, 1 Across is a three-digit square with hundreds digit 4 and units digit 1. Therefore 1 Across is $21^2 = 441$.

We can now deduce that * is 4.

FOR INVESTIGATION

9.1 Complete this crossnumber in such a way that no two clues have the same answer.

Across	**Down**
1. $3 \times$ a cube	1. $3 \times$ a square
3. $3 \times$ a square	2. $3 \times$ a square

1		2
	(shaded)	
3		

10. How many of the numbers 6, 7, 8, 9, 10 are factors of the sum $2^{2024} + 2^{2023} + 2^{2022}$?

A 1 B 2 C 3 D 4 E 5

Solution **B**

For convenience, we put $S = 2^{2024} + 2^{2023} + 2^{2022}$.

We have

$$S = 2^{2024} + 2^{2023} + 2^{2022} = 2^{2022}(2^2 + 2^1 + 1) = 2^{2022}(4 + 2 + 1) = 2^{2022} \times 7.$$

It follows that 7 is a factor of S. Also, since $8 = 2^3$, 8 is a factor of 2^{2022} and hence it is a factor of S. On the other hand, 3 and 5 are neither factors of 2^{2022} nor factors of 7. So they are not factors of S. It follows that 6 and 9 which are multiples of 3 are not factors of S. Similarly, 10 is a multiple of 5 and hence it is not a factor of S.

We therefore see that just two of the numbers 6, 7, 8, 9 and 10 are factors of S, namely 7 and 8.

For investigation

10.1 Which is the largest prime factor of $2^{2024} + 2^{2023} + 2^{2022} + 2^{2021}$?

11. Wenlu, Xander, Yasser and Zoe make the following statements:

Wenlu: "Xander is lying." Xander: " Yasser is lying."

Yasser: "Zoe is telling the truth." Zoe: "Wenlu is telling the truth."

What are the possible numbers of people telling the truth?

A 1 or 2 B 1 or 3 C 2 D 2 or 3 E 3

Solution **B**

Suppose Wenlu is telling the truth. Then Xander is lying. Therefore Yasser is telling the truth. Hence Zoe is telling the truth, and this agrees with the fact that Wenlu is telling the truth.

Hence it is possible that Wenlu is telling the truth. We have seen that in this case Wenlu, Yasser and Zoe are telling the truth.

Suppose Wenlu is lying. Then Xander is telling the truth. Therefore Yasser is lying. Hence Zoe is lying, and this agrees with the fact that Wenlu is lying.

Hence it is possible that Wenlu is lying. We have seen that in this case only Xander is telling the truth.

Therefore the number of people telling the truth is either 1 or 3.

For investigation

11.1 Wenlu says "Xander is lying", Xander say "Yasser is telling the truth", Yasser says "Zoe is lying" and Zoe says "Wenlu is telling the truth".

How many of them could be telling the truth?

12. The greatest power of 7 which is a factor of 50! is 7^k ($n! = 1\times2\times3\times4\times\ldots\times(n-1)\times n$).

What is k?

A 4 B 5 C 6 D 7 E 8

SOLUTION **E**

$50! = 1 \times 2 \times \cdots \times 50$. Because the seven numbers 7, 14, 21, 28, 35, 42 and 49 are divisible by 7, they each contribute 1 to the power of 7 which is a factor 50! In addition, because 49 is divisible by 7^2, it contributes an additional power of 7.

Therefore the highest power of 7 that divides 50! is $7 + 1 = 8$.

FOR INVESTIGATION

12.1 Find the greatest power of 3 which is a factor of 50!

The general formula for the greatest power of a prime p which is a factor of $n!$ is

$$\left\lfloor \frac{n}{p} \right\rfloor + \left\lfloor \frac{n}{p^2} \right\rfloor + \left\lfloor \frac{n}{p^3} \right\rfloor + \ldots. \qquad (1)$$

Here $\lfloor x \rfloor$ is the *integer part* of x which is defined by

$$\lfloor x \rfloor = k \Leftrightarrow k \text{ is the largest integer } \leq x.$$

For example $\left\lfloor \frac{3}{4} \right\rfloor = 0$, $\left\lfloor \frac{22}{7} \right\rfloor = 3$, and $\lfloor \sqrt{5} \rfloor = 2$.

Note that, although the sum in (1) looks infinite, whenever $p^t > n$, we have $0 < \frac{n}{p^t} < 1$ and therefore $\left\lfloor \frac{n}{p^t} \right\rfloor = 0$. Therefore we could replace (1) by the finite sum

$$\left\lfloor \frac{n}{p} \right\rfloor + \left\lfloor \frac{n}{p^2} \right\rfloor + \left\lfloor \frac{n}{p^3} \right\rfloor + \cdots + \left\lfloor \frac{n}{p^s} \right\rfloor \qquad (2)$$

where s is the largest integer such that $p^s \leq n$.

For example, since $7^3 < 1000$, but $7^4 > 1000$, it follows from (2) that the greatest power of 7 that is a factor of 1000! is

$$\left\lfloor \frac{1000}{7} \right\rfloor + \left\lfloor \frac{1000}{7^2} \right\rfloor + \left\lfloor \frac{1000}{7^3} \right\rfloor = \left\lfloor \frac{1000}{7} \right\rfloor + \left\lfloor \frac{1000}{49} \right\rfloor + \left\lfloor \frac{1000}{343} \right\rfloor = 142 + 20 + 2 = 164.$$

12.2 Find the greatest power of 2 that is a factor of 100!

12.3 Find the greatest power of 11 that is a factor of 1000!

12.4 Find the greatest power of 10 that is a factor of 1000!

12.5 Explain why the formula (1) given above for the greatest power of a prime p that divides $n!$ is correct.

13. $PQRST$ is a regular pentagon. The point U lies on ST such that $\angle QPU$ is a right angle.

What is the ratio of the interior angles in triangle PUT?

A 1 : 3 : 6 B 1 : 2 : 4 C 2 : 3 : 4 D 1 : 4 : 8 E 1 : 3 : 5

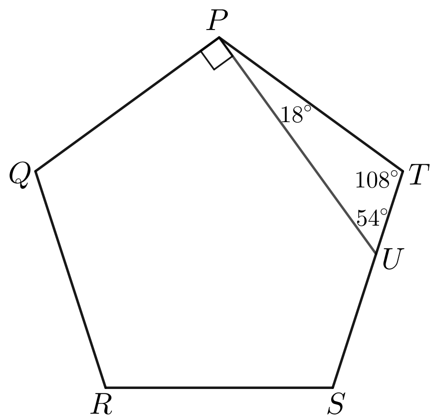
SOLUTION **A**

The interior angles of a regular pentagon are all 108°. [You are asked to prove this in Problem 13.1.] Therefore $\angle PTU = \angle QPT = 108°$.

Hence $\angle UPT = \angle QPT - \angle QPU = 108° - 90° = 18°$.

Because the angles in a triangle have sum 180°, it follows that $\angle TUP = 180° - 108° - 18° = 54°$.

Therefore the ratio of the interior angles in the triangle PUT is 18 : 54 : 108 = 1 : 3 : 6.



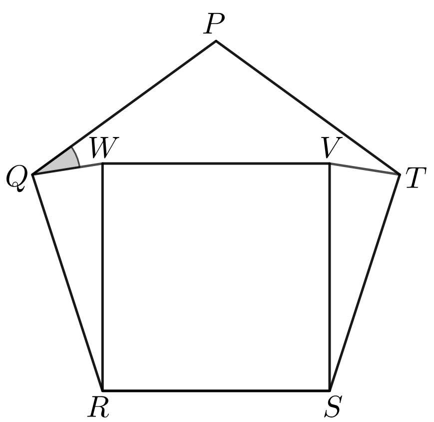
FOR INVESTIGATION

13.1 (a) Show that the sum of the interior angles of a polygon with n vertices is $(n-2)180°$.

(b) Deduce that the interior angles of a regular pentagon are equal to 108°.

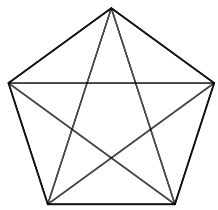
13.2 The regular pentagon $PQRST$ and the square $RSVW$ share the edge RS.

What is $\angle WQP$?



13.3 The diagram shows a regular pentagon and all its diagonals.

Find all the angles in the diagram.



14. The points P $(d, -d)$ and Q $(12 - d, 2d - 6)$ both lie on the circumference of the same circle whose centre is the origin.

What is the sum of the two possible values of d?

A -16 B -4 C 4 D 8 E 16

Solution **E**

Let O $(0, 0)$ be the origin.

Because the points P and Q lie on the same circle with centre O, we have $OP^2 = OQ^2$. That is,

$$d^2 + (-d)^2 = (12 - d)^2 + (2d - 6)^2.$$

Expanding both sides of this equation, we obtain

$$d^2 + d^2 = (144 - 24d + d^2) + (4d^2 - 24d + 36).$$

We can rearrange this equation to obtain

$$3d^2 - 48d + 180 = 0.$$

By dividing both sides of this last equation by 3, it follows that

$$d^2 - 16d + 60 = 0.$$

We can now use the fact that the sum of the roots of the quadratic equation $x^2 + px + q = 0$ is $-p$ to deduce that the sum of the two possible values of d is $-(-16)$, that is, 16.

For investigation

14.1 (a) Find the two possible values of d by solving the equation $d^2 - 16d + 60 = 0$.

(b) Hence check that the sum of the two possible values of d is 16.

14.2 Find the centre of the circle that goes through the points $(4, -14)$, $(-3, -13)$ and $(-7, -11)$.

14.3 Prove that the sum of the roots of the quadratic equation $x^2 + px + q = 0$ is $-p$.

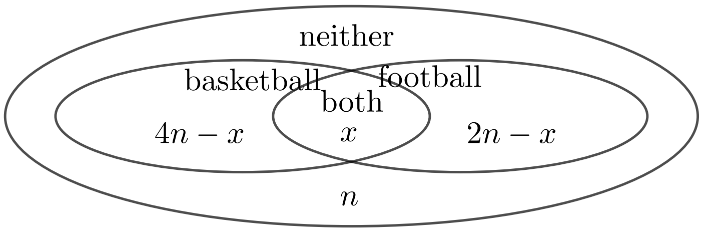
15. In Bethany's class of 30 students, twice as many people played basketball as played football. Twice as many played football as played neither.

Which of the following options could have been the number of people who played both?

A 19 B 14 C 9 D 5 E 0

Solution **D**

Let x be the number of students who played both basketball and football and let n be the number of students who played neither.



Then $2n$ students played football and hence $2 \times 2n = 4n$ students played basketball.

Hence there are $4n - x$ students who played basketball, but not football, and $2n - x$ students who played football but not basketball.

We can deduce from this that the number of students who played basketball or football or both was $(4n - x) + x + (2n - x) = 6n - x$.

Because there are 30 students in the class, the sum of the number of students who played basketball or football or both, and the number who played neither is 30. That is, $(6n-x)+n = 30$. Therefore

$$x = 7n - 30. \qquad (1)$$

The number x cannot be negative. It follows, by (1), that $4 < n$.

The number of students who played football but not basketball is $2n - x$. This number cannot be negative. Hence $x \leq 2n$. Therefore, by (1), $7n - 30 \leq 2n$. Hence $5n \leq 30$ and so $n \leq 6$.

Therefore, the only possible values of n are 5 and 6. It follows, by (1), that the only possible values of x are 5 and 12. Hence, only option D gives a possible number of students who played both.

For investigation

15.1 In Claire's class of 30 students, twice as many play neither cricket nor tennis, as play both.

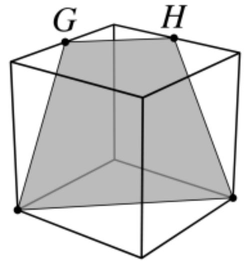
The ratio of those playing cricket to those playing tennis is 7 : 5.

How many in Claire's class play cricket?

16. G and H are midpoints of two adjacent edges of a cube. A trapezium-shaped cross-section is formed cutting through G, H and two further vertices, as shown. The edge-length of the cube is $2\sqrt{2}$.

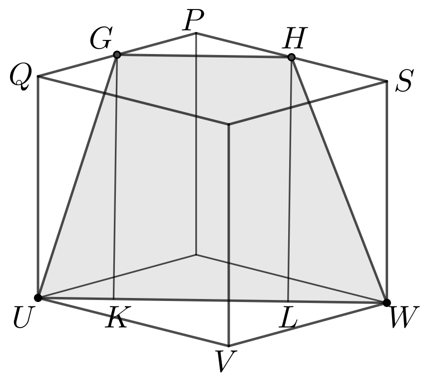
What is the area of the trapezium?

A 9 B 8 C $4\sqrt{5}$ D $4\sqrt{3}$ E $4\sqrt{2}$



SOLUTION **A**

We let P, Q, S, U, V and W be vertices of the cube, as shown in the diagram, and K, L be the feet of the perpendiculars from G, H, respectively, to UW.



We apply Pythagoras' Theorem to the right-angled triangle UVW. This gives $UW^2 = UV^2 + WV^2 = (2\sqrt{2})^2 + (2\sqrt{2})^2 = 8 + 8 = 16$. Hence $UW = 4$.

G is the midpoint of the edge PQ. Hence $PG = \sqrt{2}$. Similarly, $PH = \sqrt{2}$. Therefore, applying Pythagoras' Theorem to the right-angled triangle GPH gives $GH^2 = PG^2 + PH^2 = (\sqrt{2})^2 + (\sqrt{2})^2 = 2+2 = 4$. Hence $GH = 2$.

Also, applying Pythagoras' Theorem to the right-angled triangle GQU gives $GU^2 = GQ^2 + UQ^2 = (\sqrt{2})^2 + (2\sqrt{2})^2 = 2 + 8 = 10$. Hence $GU = \sqrt{10}$.

Since $GKLH$ is a rectangle, $KL = GH = 2$. Therefore $UK + WL = UW - KL = 4 - 2 = 2$. By symmetry, $UK = WL$. Hence $UK = WL = 1$.

Applying Pythagoras' Theorem to the right-angled triangle GKU, gives $GK^2 = GU^2 - UK^2 = (\sqrt{10})^2 - 1^2 = 10 - 1 = 9$. Hence $GK = 3$.

We now use the formula $\frac{1}{2}(a+b)h$ for the area of a trapezium whose parallel sides have lengths a and b and which has height h. [You are asked to prove this formula in Problem 16.1.]

It follows from this formula that

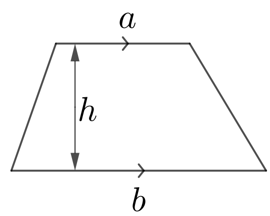
$$\text{area of the trapezium } GUWH = \tfrac{1}{2}(UW + GH)GK = \tfrac{1}{2}(4+2) \times 3 = 9.$$

FOR INVESTIGATION

16.1 Prove that the formula

$$\tfrac{1}{2}(a+b)h$$

for the area of a trapezium is correct.



17. The number $M = 124563987$ is the smallest number which uses all the non-zero digits once each and which has the property that none of the pairs of its consecutive digits makes a prime number. For example, the 5th and 6th digits of M make the number 63 which is not prime. N is the largest number which uses all the non-zero digits once each and which has the property that none of the pairs of its consecutive digits makes a prime number.

What are the 5th and 6th digits of N?

A 6 and 3 B 5 and 4 C 5 and 2 D 4 and 8 E 3 and 5

SOLUTION **E**

For convenience, in this solution by a *subnumber* of a positive number n we mean a number formed from consecutive digits of n. For example, the two-digit subnumbers of 1234 are 12, 23 and 34, and the three-digit subnumbers of 1234 are 123 and 234.

Note that this is not standard mathematical terminology, but has been introduced just for the purposes of this question.

We note first that 63 and 93 are the only numbers formed of two different non-zero digits with 3 as the unit digits that are not primes. It follows that, if the digit 3 is not the first digit of the number N, the digit 3 could only occur in N immediately after either the digit 9 or the digit 6.

We construct N by beginning with the largest digit 9, and then use all the other non-zero digits once each by always choosing the largest digit not yet used subject to the condition that 3 comes immediately after 9 or immediately after 6. In this way we obtain the number 987635421.

We now see that in the number 987635421 none of the two-digit subnumbers is a prime. Any larger number using all the digits must either begin 98765... or 98764..., but in each case the 3 would follow either 1,2,4 or 5, and so produce a two-digit subnumber that is a prime.

Therefore the largest number with the required property is $N = 987635421$. It follows that the 5th and 6th digits of N are 3 and 5.

FOR INVESTIGATION

17.1 Find the largest positive integer that uses different non-zero digits, and has the property that all its two-digit subnumbers are prime.

17.2 Find the largest positive integer that uses different non-zero digits, and has the property that all its two-digit subnumbers are divisible by 7.

17.3 Find the largest positive integer that uses different non-zero digits, and has the property that all its three-digit subnumbers are divisible by 3.

17.4 Find the largest positive integer that uses different non-zero digits, and has the property that all its three-digit subnumbers are divisible by 11.

18. How many solutions are there of the equation $1 + 2\sin X - 4\sin^2 X - 8\sin^3 X = 0$ with $0° < X < 360°$?

A 1 B 2 C 4 D 6 E 8

SOLUTION **C**

For convenience we put $x = \sin X$.

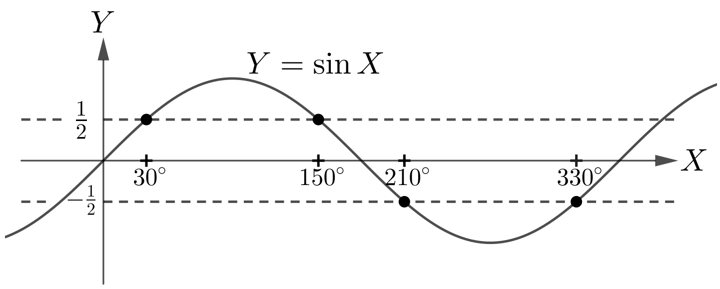
We have

$$\begin{aligned}1 + 2\sin X - 4\sin^2 X - 8\sin^3 X &= 1 + 2x - 4x^2 - 8x^3\\ &= 1 + 2x - 4x^2(1 + 2x)\\ &= (1 + 2x)(1 - 4x^2)\\ &= (1 + 2x)(1 + 2x)(1 - 2x),\end{aligned}$$

It follows that

$$\begin{aligned}&1 + 2\sin X - 4\sin^2 X - 8\sin^3 X = 0\\ &\qquad\Leftrightarrow (1 + 2x)(1 + 2x)(1 - 2x) = 0\\ &\qquad\Leftrightarrow x = -\tfrac{1}{2} \text{ or } x = \tfrac{1}{2}\\ &\qquad\Leftrightarrow \sin X = -\tfrac{1}{2} \text{ or } \sin X = \tfrac{1}{2}.\end{aligned}$$

The solutions of the equation $\sin X = -\frac{1}{2}$ with $0° < X < 360°$ are $x = 210°$ and $X = 330°$. The solutions of $\sin X = \frac{1}{2}$ in the same interval are $X = 30°$ and $X = 150°$.



Therefore the equation $1 + 2\sin X - 4\sin^2 X - 8\sin^3 X = 0$ has 4 solutions with $0° < X < 360°$.

FOR INVESTIGATION

18.1 Find all the solutions of the equation

$$10\sin^2 X - 8\sin^4 X = 3,$$

with $0° < X < 360°$.

18.2 Find all the solutions of the equation

$$\sin^4 X + \sin^2 X = 3,$$

with $0° < X < 360°$.

19. The expression $\dfrac{7n+12}{2n+3}$ takes integer values for certain integer values of n.

What is the sum of all such integer values of the expression?

A 4 B 8 C 10 D 12 E 14

SOLUTION **E**

We have

$$\frac{7n+12}{2n+3} = \frac{7}{2} + \frac{3}{2(2n+3)} = \frac{1}{2}\left(7 + \frac{3}{2n+3}\right).$$

It follows that $\dfrac{7n+12}{2n+3}$ is an integer provided that $7 + \dfrac{3}{2n+3}$ is an even integer and hence provided that $\dfrac{3}{2n+3}$ is an odd integer.

Now $\dfrac{3}{2n+3}$ is an integer provided that $2n+3$ is a factor of 3.

Therefore the only possible values of $2n+3$ are -3, -1, 1 and 3. In all these cases $\dfrac{3}{2n+3}$ is a factor of 3 and hence is odd.

When $2n+3 = -3$, we have $n = -3$ and $\dfrac{7n+12}{2n+3} = \dfrac{1}{2}\left(7 + \dfrac{3}{2n+3}\right) = \dfrac{1}{2}\left(7 + \dfrac{3}{-3}\right) = \dfrac{1}{2}\left(7 - 1\right) = 3.$

When $2n+3 = -1$, we have $n = -2$ and $\dfrac{7n+12}{2n+3} = \dfrac{1}{2}\left(7 + \dfrac{3}{2n+3}\right) = \dfrac{1}{2}\left(7 + \dfrac{3}{-1}\right) = \dfrac{1}{2}\left(7 - 3\right) = 2.$

When $2n+3 = 1$, we have $n = -1$ and $\dfrac{7n+12}{2n+3} = \dfrac{1}{2}\left(7 + \dfrac{3}{2n+3}\right) = \dfrac{1}{2}\left(7 + \dfrac{3}{1}\right) = \dfrac{1}{2}\left(7 + 3\right) = 5.$

When $2n+3 = 3$, we have $n = 0$ and $\dfrac{7n+12}{2n+3} = \dfrac{1}{2}\left(7 + \dfrac{3}{2n+3}\right) = \dfrac{1}{2}\left(7 + \dfrac{3}{3}\right) = \dfrac{1}{2}\left(7 + 1\right) = 4.$

We see that n is an integer whenever $2n+3$ is equal to either -3, -1, 1 or 3. Therefore the sum of the integer values of $\dfrac{7n+12}{2n+3}$ that correspond to integer values of n is $3 + 2 + 5 + 4 = 14$.

FOR INVESTIGATION

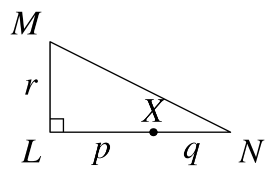
19.1 Check that $\dfrac{7n+12}{2n+3} = \dfrac{7}{2} + \dfrac{3}{2(2n+3)}$.

19.2 What is the sum of all the integer values taken by the expression

$$\frac{6n+5}{2n+9}$$

when n is an integer?

20. Triangle LMN represents a right-angled field with $LM = r$, $LX = p$ and $XN = q$. Jenny and Vicky walk at the same speed in opposite directions along the edge of the field, starting at X at the same time. Their first meeting is at M.



Which of these expressions gives q in terms of p and r?

A $\frac{p}{2} + r$ B $\sqrt{p^2 + r^2} + \frac{p}{2}$ C $\frac{pr}{2p + r}$ D $\frac{p}{2}$ E 1

Solution **C**

Because Jenny and Vicky meet after walking at the same speed, they walk the same distance. Therefore

$$XL + LM = XN + NM. \qquad (1)$$

The question tells us that $XL = p$, $LM = r$ and $XN = q$, To find NM, we apply Pythagoras' Theorem to the right-angled triangle LMN. This gives

$$NM^2 = LN^2 + LM^2 = (p + q)^2 + r^2.$$

It follows that $NM = \sqrt{(p + q)^2 + r^2}$. Substituting these values in equation (1) gives

$$p + r = q + \sqrt{(p + q)^2 + r^2}.$$

Hence

$$p + r - q = \sqrt{(p + q)^2 + r^2}. \qquad (2)$$

By squaring both sides of equation (2) we obtain

$$(p + r - q)^2 = (p + q)^2 + r^2.$$

It follows that

$$p^2 + r^2 + q^2 + 2pr - 2pq - 2rq = p^2 + 2pq + q^2 + r^2. \qquad (3)$$

Equation (3) may be rearranged to give

$$4pq + 2rq = 2pr$$

from which it follows that

$$q(2p + r) = pr.$$

Therefore

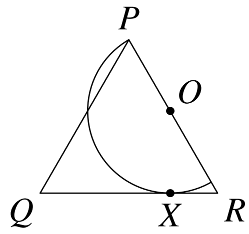
$$q = \frac{pr}{2p + r}.$$

For investigation

20.1 Check that $(p + r - q)^2 = p^2 + r^2 + q^2 + 2pr - 2pq - 2rq$.

20.2 Suppose that in this problem, $LM = LN$. In this case what is the ratio $LX : XN$?

21. Triangle PQR is equilateral. A semicircle with centre O is drawn with its diameter on PR so that one end is at P and the curved edge touches QR at X. The radius of the semicircle is $\sqrt{3}$.



What is the length of QX?

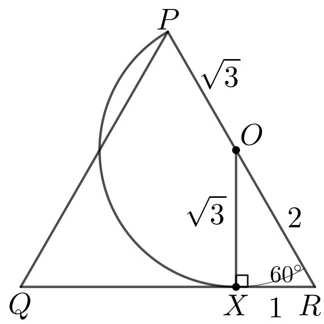
A $\sqrt{3}$ B $2 - \sqrt{3}$ C $2\sqrt{3} - 1$
D $1 + \sqrt{3}$ E $2\sqrt{3}$

Solution **D**

Because OP and OX are radii of the semicircle, we have $OP = OX = \sqrt{3}$.

Because the semicircle touches QR at X, the line QR is a tangent to the semicircle at X and therefore the radius OX is perpendicular to QR. Therefore OXR is a right-angled triangle.

Because the triangle PQR is equilateral, $\angle ORX = 60°$.



Hence

$$\frac{OX}{OR} = \sin 60° = \frac{\sqrt{3}}{2} \quad \text{and} \quad \frac{XR}{OR} = \cos 60° = \frac{1}{2}.$$

It follows that

$$OR = \frac{2}{\sqrt{3}} \times OX = \frac{2}{\sqrt{3}} \times \sqrt{3} = 2,$$

and, therefore,

$$XR = \frac{1}{2} \times OR = \frac{1}{2} \times 2 = 1.$$

We can now deduce that

$$QR = PR = OR + OP = 2 + \sqrt{3}.$$

Therefore

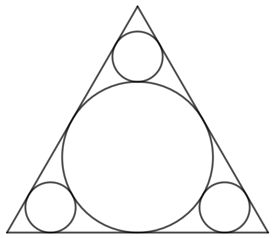
$$QX = QR - XR = (2 + \sqrt{3}) - 1 = 1 + \sqrt{3}.$$

For investigation

21.1 The diagram shows an equilateral triangle with sides of length 1, and four circles.

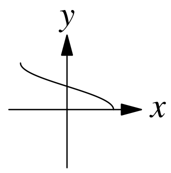
The largest circle touches the three sides of the triangle. [It is the *incircle* of the triangle.]

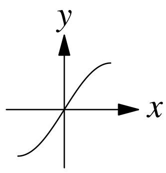
Each of the smaller circles touches two sides of the triangle and the largest circle.

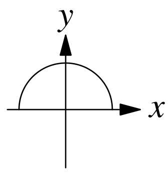


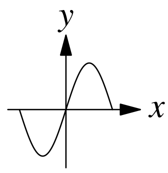
Find the radius of the smaller circles.

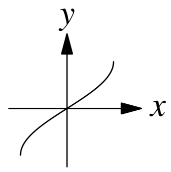
22. Which diagram could be a sketch of the curve $y = \sin(\cos^{-1} x)$?

A


B


C


D


E


Solution **C**

Our first method is to find points that lie on the curve, and to note that these points are on just one of the curves given as options. In the context of the SMC it should be safe to assume that the question setters have not made a mistake and therefore this one curve is the correct one.

However, this is not a fully justified mathematical answer. In the second method we show that the equation of the curve may be written in a more familiar way. This enables us to deduce which is the correct option.

Method 1

Since $\cos^{-1} 0 = 90°$, when $x = 0$ we have $y = \sin(\cos^{-1} 0) = \sin 90° = 1$. It follows that the point $(0, 1)$ is on the curve. This rules out all the options other than A and C.

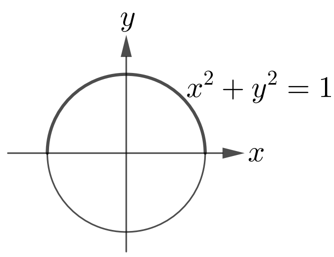
Since $\cos^{-1}(-1) = 180°$, when $x = -1$ we have $y = \sin(\cos^{-1}(-1)) = \sin 180° = 0$. It follows that the point $(-1, 0)$ is on the curve. This rules out option A.

We conclude that just option C could be a sketch of the curve.

Method 2

If we substitute $\cos^{-1} x$ for θ in the identity $\cos^2 \theta + \sin^2 \theta = 1$, we obtain $\cos^2(\cos^{-1} x) + \sin^2(\cos^{-1} x) = 1$. (1)

We are given that $y = \sin(\cos^{-1} x)$. Also, because $\cos^{-1}$ is the inverse of the function cos, $x = \cos(\cos^{-1} x)$. Hence it follows from (1) that $x^2 + y^2 = 1$. This is the equation of the circle with centre $(0, 0)$ and radius 1. We leave it to the reader [See Problem 22.1] to work out why the curve is just the top half of this circle.



It follows that just option C could be a sketch of the curve.

For investigation

22.1 Explain why the curve corresponding to the equation $y = \sin(\cos^{-1} x)$ is just the part of the circle with equation $x^2 + y^2 = 1$ where $y \geq 0$.

23. The length of a rectangular piece of paper is three times its width. The paper is folded so that one vertex lies on top of the opposite vertex, thus forming a pentagonal shape.

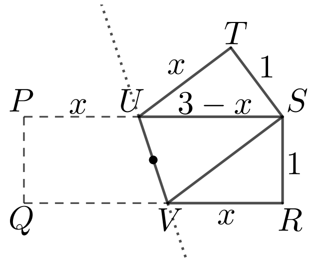
What is the area of the pentagon as a fraction of the area of the original rectangle?

A $\frac{2}{3}$ B $\frac{11}{16}$ C $\frac{12}{17}$ D $\frac{13}{18}$ E $\frac{14}{19}$

SOLUTION **D**

Let P, Q, R and S be the vertices of the rectangular piece of paper. We choose units so that $PQ = 1$ and $PS = 3$.

Because the fold moves Q to S, the fold line goes through the midpoint of QS which is the centre of the rectangle. We let U and V be the points where this fold line meets the edges of the rectangle, as shown in the diagram.



We let x be the length of VR. The rectangle has a rotational symmetry that interchanges U and V. Therefore, $PU = VR = x$. Hence $US = 3 - x$.

We let T be the point where the vertex Q ends up after the fold. The quadrilateral $STUV$ is the reflection of $QPUV$ in the line through U and V. It follows that $TS = 1$, $UT = x$ and $\angle UTS$ is a right angle.

We need to find the area of the pentagon $RSTUV$. This pentagon is made up of the trapezium $RSUV$ and the triangle STU.

It follows from the rotational symmetry of the rectangle that the area of the trapezium $RSUV$ is half the area of the rectangle $PQRS$. Therefore the area of the trapezium is $\frac{1}{2}\left(3 \times 1\right) = \frac{3}{2}$.

The area of the right-angled triangle STU is $\frac{1}{2}\left(1 \times x\right) = \frac{x}{2}$.

By Pythagoras' Theorem applied to the triangle UST, we have $1 + x^2 = (3 - x)^2$. Therefore $1 + x^2 = 9 - 6x + x^2$. It follows that $6x = 8$. Hence $x = \frac{8}{6} = \frac{4}{3}$.

It follows that the area of the pentagon $RSTUV$ is given by $\frac{3}{2} + \frac{4/3}{2} = \frac{3}{2} + \frac{2}{3} = \frac{13}{6}$.

The area of rectangle $PQRS$ is 3. Therefore the area of the pentagon as a fraction of the area of the original rectangle is $\frac{13/6}{3} = \frac{13}{18}$.

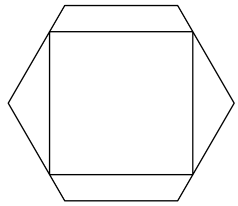
FOR INVESTIGATION

23.1 Consider the similar problem with an A4 sheet of paper where the ratio of the length to the width is $\sqrt{2} : 1$. What is the area of the pentagon as a fraction of the area of the original rectangle in this case?

23.2 In a similar problem the length of a rectangular piece of paper is k times its width. After the paper is folded so that one vertex lies on top of the opposite vertex, the area of the pentagon that is formed is 74% of the area of the rectangle. What is the value of k?

24. A square has its vertices on the edges of a regular hexagon. Two of the edges of the square are parallel to two edges of the hexagon, as shown in the diagram. The sides of the hexagon have length 1.

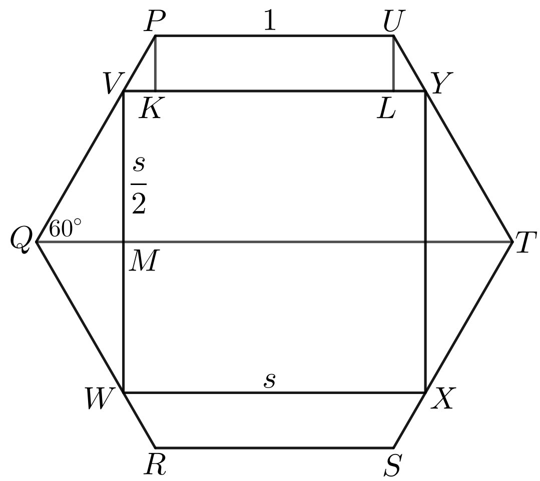
What is the length of the sides of the square?



A $\frac{5}{4}$ B $3 - \sqrt{3}$ C $\frac{4}{3}$ D $\sqrt{2}$ E $\frac{3}{2}$

SOLUTION **B**

We label the vertices of the hexagon P, Q, R, S, T and U, and the vertices of the square V, W, X and Y, as shown in the diagram.



Also, we let K and L be the feet of the perpendiculars from P and U, respectively, to VY, and we let M be the point where VW meets QT.

We let the length of the sides of the square be s.

The internal angles of the regular hexagon are 120°. The figure is symmetrical about the line QT. Therefore $\angle VQM = 60°$, $\angle VMQ = 90°$ and $VM = \frac{s}{2}$.

From the right-angled triangle VMQ, we have $\frac{VM}{VQ} = \sin 60° = \frac{\sqrt{3}}{2}$. Therefore $VQ = \frac{2}{\sqrt{3}}VM = \frac{2}{\sqrt{3}} \times \frac{s}{2} = \frac{s}{\sqrt{3}}$.

$KLUP$ is a rectangle. Therefore $KL = PU = 1$. Therefore, $VK + LY = VY - KL = s - 1$. By symmetry, $VK = LY$. Hence $VK = \frac{s-1}{2}$.

$\angle PKV = 90°$ and $\angle PVK = 60°$. Therefore in the triangle PVK we have $\frac{VK}{PV} = \cos 60° = \frac{1}{2}$. Hence $PV = 2VK = s - 1$.

Since $PQ = 1$, we have $PV + VQ = 1$, and hence $s - 1 + \frac{s}{\sqrt{3}} = 1$. Hence $s\left(1 + \frac{1}{\sqrt{3}}\right) = 2$. That is, $s\left(\frac{\sqrt{3}+1}{\sqrt{3}}\right) = 2$. Therefore $s = \frac{2\sqrt{3}}{\sqrt{3}+1} = \frac{2\sqrt{3}(\sqrt{3}-1))}{(\sqrt{3}+1)(\sqrt{3}-1)} = \frac{6-2\sqrt{3}}{2} = 3 - \sqrt{3}$.

FOR INVESTIGATION

24.1 Express the value of $\dfrac{\text{area of the square } VWXY}{\text{area of the hexagon } PQRSTU}$ in the form $a + b\sqrt{3}$, where a and b are rational numbers.

25. What is the area of the part of the xy-plane within which $x^3y^2 - x^2y^2 - xy^4 + xy^3 \geq 0$ and $0 \leq x \leq y$?

A $\frac{1}{4}$ B $\frac{1}{2}$ C 1 D 2 E 4

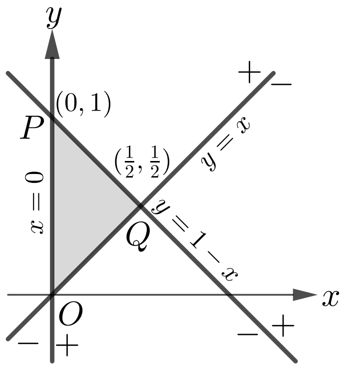
SOLUTION **A**

We have

$$\begin{aligned} x^3y^2 - x^2y^2 - xy^4 + xy^3 &= xy^2(x^2 - x - y^2 + y) \\ &= xy^2((x^2 - y^2) - (x - y)) \\ &= xy^2((x - y)(x + y) - (x - y)) \\ &= xy^2(x - y)(x + y - 1). \end{aligned}$$

In the region where $0 \leq x \leq y$, we have $xy^2 \geq 0$ and $x - y \leq 0$. Therefore in this region $xy^2(x - y)(x + y - 1) \geq 0 \Leftrightarrow (x + y - 1) \leq 0$.

Therefore the region of the xy-plane within which we have $x^3y^2 - x^2y^2 - xy^4 + xy^3 \geq 0$ and $0 \leq x \leq y$ is the region where $x + y - 1 \leq 0$ and $0 \leq x \leq y$.



The line with equation $x = 0$ (the y-axis) divides the plane into the region where $x > 0$ and the region where $x < 0$. In the diagram the symbol + is used to indicate the side of the line where $x > 0$ and − to indicate the side where $x < 0$.

In a similar way the line with equation $y = x$ divides the plane into the region where $y > x$ and the region where $y < x$. Again, we use + and − to mark these two regions.

Also, the line with the equation $x + y - 1 = 0$ (or, equivalently, $y = 1 - x$) divides the plane into the region where $x + y - 1 > 0$ and the region where $x + y - 1 < 0$.

Therefore the region whose area we need to find is the triangle OPQ which is bounded by the lines with equations $x = 0$, $x = y$ and $y = 1 - x$, as shown in the diagram.

The point O is the origin where the lines $x = 0$ and $y = x$ meet. The point P is the point where the lines $x = 0$ and $y = 1 - x$ meet. The co-ordinates of P are $(0, 1)$. The point Q is the point where the lines $y = x$ and $y = 1 - x$ meet. The co-ordinates of Q are $(\frac{1}{2}, \frac{1}{2})$.

The triangle OPQ has a base OP which has length 1. The height of the triangle with this base is the distance of Q from the y-axis which is $\frac{1}{2}$. Therefore, the area of this triangle is

$$\frac{1}{2}\left(1 \times \frac{1}{2}\right) = \frac{1}{4}.$$

FOR INVESTIGATION

25.1 What is the area of the part of the xy-plane where $x^2 + y^2 \leq 4$ and $x^2 - y^2 \leq 0$?

25.2 What is the area of the part of the xy-plane where $x^3y^2 - xy^4 \leq 0$, $-2 \leq x \leq 2$ and $-3 \leq y \leq 3$?