UK Maths Trust

Andrew Jobbings Senior Kangaroo

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Solutions

These are polished solutions and do not illustrate the process of failed ideas and rough work by which candidates may arrive at their own solutions.

It is not intended that these solutions should be thought of as the 'best' possible solutions and the ideas of readers may be equally meritorious.

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1. A particular flag is in the shape of a rectangle divided into five smaller congruent rectangles as shown. When written in its lowest terms, the ratio of the side lengths of the smaller rectangle is $\lambda : 1$, where $\lambda < 1$.
What is the value of 360λ?

SOLUTION **120**

Let a, b be the side lengths of the smaller rectangle, with $a < b$.
We observe that $b = 3a$, so $\lambda = \frac{1}{3}$.
The value of 360λ is $360 \times \frac{1}{3} = 120$.

2. The integers m and n satisfy the equation $3^m \times n = 7! + 8! + 9!$.
What is the smallest possible value for n?

SOLUTION **560**

$7! + 8! + 9! = 7!(1 + 8 + 8 \times 9) = 7! \times 81 =$
$1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 3^4 = 1 \times 2 \times 3 \times 4 \times 5 \times 2 \times 3 \times 7 \times 3^4 = 1 \times 2 \times 4 \times 5 \times 2 \times 7 \times 3^6$.
The smallest value of n corresponds with $m = 6$, and is $1 \times 2 \times 4 \times 5 \times 2 \times 7 = 560$.

3. How many of the integers from 2^{10} to 2^{18} inclusive are divisible by 2^9?

SOLUTION **511**

We observe that $2^{10} = 2 \times 2^9$, and $2^{18} = 512 \times 2^9$. Therefore, on multiplying 2^9 by any integer between 2 and 512 inclusive, we obtain an integer between 2^{10} and 2^{18} inclusive. There are $512 - 1 = 511$ such integers.

4. Months of the year are usually labelled numerically by '01' for January, '02' for February, and so on, through to '12' for December. Lydia notices that during January, the number of letters in the name of the month is greater than the month's numerical label (i.e. $7 > 1$).
For how many days during 2024 will the date have that property?

SOLUTION **121**

The only months that have the property described are January ('01'), February ('02'), March ('03'), and April ('04'). During 2024, these will have $31 + 29 + 31 + 30 = 121$ days in total.

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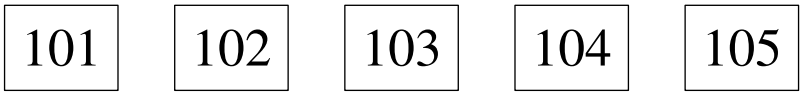
5. The number C is defined as the sum of all the positive integers n such that $n - 6$ is the second largest factor of n. What is the value of $11C$?

SOLUTION **308**

Any integer is at most double its second largest factor. Therefore, we need only test the integers up to and including $n = 12$ (which is double $n - 6 = 6$) to see if each has $n - 6$ as its second largest factor. Of these, only 7, 9 and 12 do. These have total $C = 7 + 9 + 12 = 28$, so $11C = 11 \times 28 = 308$.

6. Five cards have the numbers 101, 102, 103, 104 and 105 on their fronts.



On the reverse, each card has a statement printed as follows:
101: The statement on card 102 is false
102: Exactly two of these cards have true statements
103: Four of these cards have false statements
104: The statement on card 101 is false
105: The statements on cards 102 and 104 are both false
What is the total of the numbers shown on the front of the cards with TRUE statements?

SOLUTION **206**

If card 101 is true then both card 102 is false and card 104 is false. So card 105 is true and in turn card 103 is false. This means we have exactly two true cards but that simultaneously card 102 is false. This is a contradiction.
So card 101 must be false. This means both card 104 is true and card 102 is true. This means card 105 is false and card 103 is false. This is two true cards.
The sum of the numbers on the two true cards is $102 + 104 = 206$.

7. A container in the shape of a rectangular box is partially filled with $120\,\text{m}^3$ of water. The depth of the water is either 2 m, 3 m or 5 m, depending on which side of the box is on the ground.
What is the volume of the container in m^3?

SOLUTION **240**

Let the dimensions of the box in metres be x, y, z, with $x < y < z$. We know $5xy = 3xz = 2yz = 120$. Therefore, $5xy \times 3xz \times 2yz = 120^3$, giving equation $30x^2y^2z^2 = 120^3$.
We deduce that $x^2y^2z^2 = \frac{120 \times 120 \times 120}{30} = 4 \times 120 \times 120$.
Therefore, the volume, in m^3, of the container is $xyz = 2 \times 120 = 240$.

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8. The product of all the factors of 5^{15} is 5^P.
What is the value of P?

SOLUTION **120**

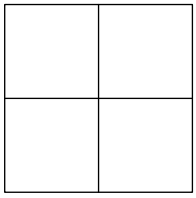
The factors of 5^{15} are $5^0, 5^1, 5^2, \cdots, 5^{15}$.
Their product is $5^{1+2+\cdots+15} = 5^{120}$.

9. What is the lowest prime number that is thirteen more than a cube?

SOLUTION **229**

We consider the cube numbers in sequence until we find one that is thirteen less than a prime.
$1 + 13 = 14$, $8 + 13 = 21$, $27 + 13 = 40$, $64 + 13 = 77$, $125 + 13 = 138$, none of which are prime.
$216 + 13 = 229$, which is prime, and therefore 229 is the answer.

10. The smallest four two-digit primes are written in different squares of a 2×2 table.
The sums of the numbers in each row and column are calculated.
Two of these sums are 24 and 28.
The other two sums are c and d, where $c < d$.
What is the value of $5c + 7d$?



SOLUTION **412**

The only way of obtaining sums of 24 and 28 are using $11 + 13 = 24$ and $11 + 17 = 28$. One possible configuration is shown in the diagram, with other two sums $c = 13 + 19 = 32$ and $d = 17 + 19 = 36$. The value of $5c + 7d$ is $5 \times 32 + 7 \times 36 = 412$.

11	13
17	19

11. The town of Coppersville uses some special coins. Edward has five 5 p coins, four 4 p coins, three 3 p coins, two 2 p coins and one 1 p coin. The smallest amount he cannot pay with his coins alone is W p.
What is the value of W?

SOLUTION **056**

Edward has $5 \times 5 + 4 \times 4 + 3 \times 3 + 2 \times 2 + 1 \times 1$ =55 p in total. He can pay amounts from 1 p to 5 p using the 1 p and 2 p coins only. He can pay amounts from 6 p to 14 p by adding one, two, or three 3 p coins to his payments for amounts 1 p to 5 p. He can pay amounts from 15 p to 30 p by adding one, two, three, or four 4 p coins to his payments for amounts 1 p to 14 p. He can pay amounts from 31 p to 55 p by adding one, two, three, four, or five 5 p coins to his payments for amounts 1 p to 30 p. Therefore, he can pay any amount up to and including 55 p, but cannot pay 56 p.

12. How many positive integers are there whose digits do not include 0, and whose digits have sum 6?

SOLUTION **032**

Integers with digit sum 6 may be grouped according to their digits:
111111;
21111, 12111, 11211, 11121, 11112;
3111, 1311, 1131, 1113;
411, 141, 114;
15, 51;
6;
2211, 2121, 2112, 1221, 1212, 1122;
222;
321, 312, 231, 213, 132, 123;
33;
42, 24.
There are 32 such integers.

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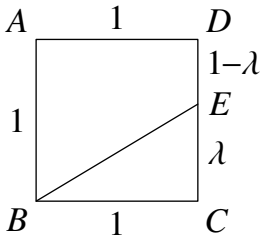
13. On a square $ABCD$ a line segment BE is drawn such that the point E lies on the side CD. The perimeter of triangle BCE is three-quarters of the perimeter of the square $ABCD$. The ratio of lengths $CE : CD$ is $\lambda : 1$.
What is the value of $960 \times \lambda$?

SOLUTION **720**

Let the square's side have length 1. The triangle's hypotenuse has length $\sqrt{1+\lambda^2}$, so the triangle's perimeter is $1 + \lambda + \sqrt{1+\lambda^2} = 3$. Therefore, $\sqrt{1+\lambda^2} = 2 - \lambda$ from which we obtain the equation $1 + \lambda^2 = 4 - 4\lambda + \lambda^2$.
Solving this equation we find $\lambda = \frac{3}{4}$, so $960\lambda = 720$.



14. Three different numbers are chosen at random from the list 1, 3, 5, 7, 9, 11, 13, 15, 17, 19. The probability that one of them is the mean of the other two is p.
What is the value of $\frac{120}{p}$?

SOLUTION **720**

There are $\frac{10\times9\times8}{3\times2\times1} = 120$ equally likely choices of three numbers.
It is not possible for 1 or 19 to be the mean.
For 3 to be the mean there is only one possible selection, namely $1, 3, 5$.
Similarly 17 can only be the mean if $15, 17, 19$ is selected.
5 is the mean of both $1, 9$ and $3, 7$; while 15 is the mean of both $11, 19$ and $13, 17$.
7 is the mean of $1, 13$; $3, 11$; and $5, 9$; while 13 is the mean of $7, 19$; $9, 17$; and $11, 15$.
Finally, 9 is the mean of $1, 17$; $3, 15$; $5, 13$; and $7, 11$; while 11 is the mean of $3, 19$; $5, 17$; $7, 15$; and $9, 13$.
There are 20 such selections in total, so $p = \frac{20}{120} = \frac{1}{6}$. Therefore, $\frac{120}{p} = 120 \times 6 = 720$.

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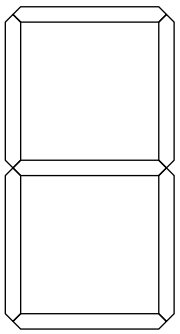
15. Kiran is designing a game which involves a bag of twenty-one marbles. Some of the marbles are blue, the rest are red. To play the game, two marbles are drawn out. The game is won if at least one red marble is drawn. To ensure the probability of the game being won is exactly one-half, Kiran uses B blue marbles and R red marbles.
What is the value of $B^2 + R^2$?

SOLUTION **261**

The probability of selecting blue then blue is 0.5.
Therefore, $\frac{B}{21} \times \frac{B-1}{20} = 0.5$.
This leads to the quadratic equation $B^2 - B - 210 = 0$.
This factorises as $(B - 15)(B + 14) = 0$ with $B = 15$ as the only positive solution.
This means $B = 15$ and $R = 6$, so $B^2 + R^2 = 225 + 36 = 261$.

16. Stephen's calculator displays only one digit, as shown in the diagram. Unfortunately, the calculator is broken. Each time he switches it on, each of the seven bars will either illuminate (show up) or not, with probability 0.5. The resultant display correctly shows one of the ten digits 0 - 9 with probability $\frac{a}{b}$.
Given that $\frac{a}{b}$ is written in its lowest terms, what is the value of $9a + 2b$?



SOLUTION **173**

As each bar illuminates with probability 0.5 then each of the 2^7 possible displays occur with equal likelihood. There are ten displays which correctly show a digit (namely, those which show the digits 0-9). The probability of obtaining one of these displays is $\frac{10}{2^7} = \frac{10}{128}$ which is $\frac{5}{64}$ in lowest terms. So $a = 5$ and $b = 64$. The value of $9a + 2b$ is $9 \times 5 + 2 \times 64 = 45 + 128 = 173$.

17. The integer part of $\sqrt{72 + \sqrt{72 + \sqrt{72 + \sqrt{72 + \sqrt{72 + \sqrt{72}}}}}}$ is A.
What is the value of $50A$?

SOLUTION **400**

The expression is clearly greater than $\sqrt{64}$, so $A \geq 8$.
By replacing the rightmost '72' in the expression with 81, we obtain the slightly larger expression

$$\sqrt{72 + \sqrt{72 + \sqrt{72 + \sqrt{72 + \sqrt{72 + \sqrt{81}}}}}} = \sqrt{72 + \sqrt{72 + \sqrt{72 + \sqrt{72 + \sqrt{81}}}}} =$$

$$\sqrt{72 + \sqrt{72 + \sqrt{72 + \sqrt{81}}}} = \sqrt{72 + \sqrt{72 + \sqrt{81}}} = \sqrt{72 + \sqrt{81}} = \sqrt{81} = 9.$$

Therefore, $A < 9$. Since $8 \leq A < 9$ we know $A = 8$ and $50A = 400$.

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18. The number 2024 may be split into its first two digits and its last two digits to form the numbers 20 and 24. The highest common factor of these numbers, HCF(20,24) is equal to 4. Similarly, 2025 may be split, and HCF(20,25)=5.
For how many remaining years this century (i.e. after 2025 and up to and including 2099) will this highest common factor be equal to one?

SOLUTION **030**

Since $20 = 2 \times 2 \times 5$ any year with last two digits 'YY' that form neither an even number nor a multiple of five will have $HCF(20, YY) = 1$.
That is, any year ending in 1, 3, 7, or 9.
This decade there are two such years remaining (namely 2027 and 2029), while the remaining seven decades of this century have four such years each.
Therefore, there are $2 + 7 \times 4 = 30$ such years.

19. Next year's 'leap day', 29th February 2024, may be written in 'ddmmyy' format as the number 290224. What is the sum of all the two-digit factors of 290224?

SOLUTION **397**

The prime factorisation of 290224 is $290224 = 2 \times 2 \times 2 \times 2 \times 11 \times 17 \times 97$.
Since the product of any two of 11, $16 = 2^4$, 17 and 97 will not be a two-digit number, we need only consider 16; as well as 11, 17, 97 and any multiples of each of these by the powers of two.
The only such two-digit multiples are 22, 44, 88; 34, 68.
The sum of these numbers is $11 + 16 + 17 + 97 + 22 + 44 + 88 + 34 + 68 = 397$.

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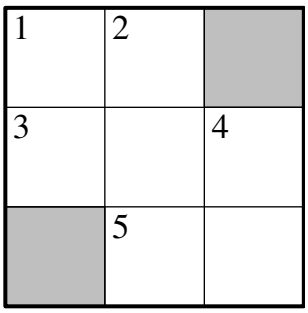
20. Each cell in this cross-number can be filled with a non-zero digit so that all of the conditions in the clues are satisfied.



The digits used are not necessarily distinct.

ACROSS
1. Four less than a factor of 105.
3. One more than a palindrome.
5. The square-root of the answer to this Kangaroo question.

DOWN
1. Two less than a square.
2. Four hundred less than a cube.
4. Six less than the sum of the answers to two of the other clues.

What is the square of the answer to 5 ACROSS?

SOLUTION **841**

The two-digit factors of 105 are 15, 21 and 35, so 1 ACROSS is one of 11, 17 or 31.
However, as 2 DOWN is 400 less than a cube it cannot start with a 7, so 1 ACROSS is either 11 or 31.
If 1 ACROSS is 11 then 1 DOWN is 14, 2 DOWN is 112 and 3 ACROSS is 415. This means 4 DOWN is at least 50 which is too large to satisfy its clue, since 5 ACROSS is at most 29.
So 1 ACROSS is 31. This means 1 DOWN is 34, 2 DOWN is 112 and 3 ACROSS is 415.
Checking each of the digits 1 to 9 in turn for the units digit of 4 DOWN yields 9 as the only possibility, with $59 = 31 + 34 - 6$.
Therefore, 5 ACROSS is 29 and its square is 841.

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