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Maclaurin 2023 Markers' Report

Both candidates and their teachers will find it helpful to know something of the general principles involved in marking Olympiad-type papers. The preliminary paragraphs therefore serve as an exposition of the 'philosophy' which has guided both the setting and marking of all such papers at all age levels, both nationally and internationally.

What we are looking for, essentially, is solutions to problems. This approach is therefore rather different from what happens in public examinations such as GCSE and A level, where credit is given for the ability to carry out individual techniques regardless of how these techniques fit into a protracted argument. Such marking is cumulative; a candidate may gain 60% of the available marks without necessarily having a clue about how to solve the final problem. Indeed, the questions are generally structured in such a way as to facilitate this approach, divided into many parts and not requiring an overall strategy for tackling a multi-stage argument.

In distinction to this, Olympiad-style problems are marked by looking at each question synoptically and deciding whether the candidate has some sort of overall strategy or not. An answer which is essentially a solution, but might contain either errors of calculation, flaws in logic, omission of cases or technical faults, will be marked on a '10 minus' basis. One question we often ask is: if we were to have the benefit of a two-minute interview with this candidate, could they correct the error or fill the gap? On the other hand, an answer which shows no sign of being a genuine solution is marked on a '0-plus' basis; up to 3 marks might be awarded for particular cases or insights. It is therefore important that candidates taking these papers realise the importance of the rubric about trying to finish whole questions rather than attempting lots of disconnected parts.

Q1

This question, which involves a ladder placed against a wall in various positions, was solved confidently by many candidates. In all three positions, the theorem of Pythagoras could be used to relate the length of the ladder to the distances of the top and bottom from the corner.

It is sensible to begin by defining some variables. For example, you might define the length of the ladder, measured in cm, as l, and the initial distances, measured in cm, of the top and bottom of the ladder from the corner as a and b respectively. This gives rise to the two equalities

$a^2 + b^2 = (a - 8)^2 + (b + 4)^2 = (a - 17)^2 + (b + 7)^2$

and most candidates managed this confidently. However, this was only worth 3 marks, since candidates had to carry out some algebraic manipulation to earn a mark of 7 or more. Note, incidentally, that units such as centimetres should not appear in algebraic equations.

The usual first step was to expand the brackets and simplify by cancelling the terms a^2 and b^2 to produce these simultaneous equations.

$2a - b = 10$

$17a - 7b = 169$

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These are then solved to obtain solutions $a = 33$ and $b = 56$. It is then necessary to calculate the length of the ladder which is 65 cm. A mark was deducted for forgetting to do this; it was, after all, required by the question.
It is sensible to check your answers, obtaining (once common factors are cancelled) the three Pythagorean triples (3, 4,5), (5,12,13) and (33,56, 65). Candidates who made careless errors were credited for their correct approach to the problem.

Q2

This is a problem about number theory, and, in particular, the rules for deciding whether a positive integer is a multiple of 2, 3, 4, 5 or 6. These rules are then applied to the situation described in the question. Most candidates understood how the grid worked, and particularly that you had to read the three-digit numbers clockwise in the diagram.
It turns out that the cases 2, 3 and 5 are easy. For $k = 2$, it is enough to place the four even digits in the corners of the grid. However, $k = 5$ is impossible since it would require a 5 in each corner, and there is only one available. The case $k = 3$ also turns out to be possible, and the best way to show this is to give an example. It is useful (but not essential for this case) to know the divisibility rule for 3, namely that the sum of the digits of a number is divisible by 3. A few candidates stated this rule but did not actually produce an example, for which they were penalised. Most candidates managed 3 marks for these three cases.
The case $k = 4$ is more difficult, largely because it is impossible. Showing that something cannot be done is usually harder than showing that it can, since it is not enough just to say that you tried hard and did not get anywhere. The first insight is to realise that a three-digit number is divisible by 4 if, and only if, the number consisting of the final two digits are divisible by 4. This is because 100 is divisible by 4. Now we clearly need 2, 4, 6 and 8 to be at the corners, so this leaves 1, 3, 5 and 7 for the tens digits. However, none of the numbers 14, 34, 54 or 74 is a multiple of 4, so this cannot be done. In the mark-scheme (which is rather unusual since it is 'additive') 2 marks were available for this case.
This leaves the case k =6 and it was worth 5 marks. A number is divisible by 6 if, and only if, it is divisible by 2 and 3. A mark was awarded for this observation. The four even digits must be in the corners, which leaves the four odd digits to go in the available spaces. At this point there were quite a lot of essays trying to account for all possible ways of filling the cells and showing that none of them worked. These attempts proved hard to mark and were assessed on their own merits. However, it is a good idea to add together the digit sums for the four numbers, which is
$1 + 3 + 5 + 7 + 2(2 + 4 + 6 + 8)$, since the corners are counted twice. This turns out to be 56, which is not a multiple of 3. Hence it is impossible to achieve the case $k = 6$.
Note, by the way, that if this calculation had yielded a multiple of 3, that would not have been enough to show that it was possible to fill the grid. Again, an example would be needed.

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Q3

There are many ways of tackling this geometry problem. Most solutions from candidates used a sequence of calculations, often using Pythagoras. There are also elegant and short methods relying on the identification of similar triangles.

However, most such approaches used the fact that the point of tangency Y is the midpoint of AB. This was sometimes assumed without any mention of the fact, or it was stated being obvious. A more sophisticated version of this is to begin by drawing a circle tangential to AB at its midpoint, and then assuming that the antipodal point on the circle lies on AD produced. In either case, this assumption is a serious flaw and was penalised heavily.

It is actually very easy to show that Y is the midpoint of AB – for example, by congruent triangles or by projection – but the rubric of the competition is clear on the fact that all mathematical statements must be justified. Hence the mark scheme stated that any student who made such assumptions about the diagram without proving them would achieve a maximum of 5 marks.

On this basis, deductions (from 5 or 10) would be made for inadequate manipulation or partial arguments. For example, obtaining the correct ratio for $AX:AD$ as $1:4$ but failing to deduce that $AX:XD = 1:3$ would lead to a deduction of 1 mark.

Q4

This was a fairly straightforward problem which relies on the fact that the equations involved have solutions which are positive integers. Such Diophantine equations, to give them their proper name, often crop up in Maclaurin papers.

It was sensible to begin by defining variables r, y and b for the numbers of beads of the appropriate colours. From this an equation such as $y^2 = r(30 - r)$ is obtained by eliminating b . It is sensible to convert a quadratic expression in r into completed square form so r appears in only one term rather than two. Then this in turn can be manipulated into a form with an algebraic expression on one side and a number on the other, such as $(r + 15)^2 - y^2 = 225$. A further rearrangement using the difference of squares leads to this useful form:

$(r + 15 + y)(r + 15 - y) = 225$

The two expressions on the left represent positive integers whose product is 225. This can only happen in a finite number of ways, and it is relatively easy, perhaps using a tabular form, to deduce that there are exactly four solutions.

The threshold for 7 marks was achieving an equation such as one above. Various deductions from 10 were made for missing solutions or for other careless errors. Candidates who did not reach the threshold could obtain marks up to a maximum of 3 for expressing the problem algebraically, eliminating a variable and finding solutions by inspection.

Markers observed that some candidates tried solving equations without realising that the solutions had to be integers, and others hit a hurdle by failing to use difference of two squares. It was also sensible, if not perhaps strictly necessary, to check your answers.

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Q5

This combinatorial problem is about colouring a 4 by 4 grid with four colours under the condition that each 2 by 2 sub-square contains one square of each colour. The aim is to prove that the corner squares must all be the same colour.

There are two principal types of solution. One is by contradiction: we suppose that the corner squares are not the same colour and show that this is impossible without breaking the rules. The other is, by some sort of counting argument, to show that there is no choice but to colour the corners differently.

The rule about each sub-square has a fairly obvious consequence. Call two adjacent squares a 'domino'. Then dominos 'replicate' in the grid, in the sense that if there are three adjacent parallel dominos, the two outer ones must share the same two colours – although not necessarily in the same order. This fact is crucial to any solution of the problem. It is, by the way, possible to write out either type of approach without mentioning dominos explicitly, and that is done in the model solutions.

The contradiction approach proved to be much more popular than the construction. It seems sensible to begin by colouring the top left square red, say, and showing that none of the other corners can be red. It is then quite straightforward to show that neither of the adjacent corners can be red. It is a little harder to show that the opposite corner cannot be red, but if we trace dominos around the grid, this eventually works. The key was to check all cases had been covered.

The construction approach is different. It does not consist in trying to enumerate all possible gridfills, although it is possible to deduce this in the process. A good start is to start by shading the central sub-square and show what happens in the four dominos on its fringes. It turns out that we will have used each of four colours three times. However, the completed grid will employ each colour four times, and so the four corner squares are of different colours.

A more advanced approach than this involved counting how many sub squares each shaded square is in. By noting that each colour needed to be in nine sub squares, but all squares are in an even number of sub squares except the corners, it was easy to deduce each colour must be in a corner.

These approaches can lead to more general solutions about what would happen for larger rectangles satisfying the conditions of the question.

If a candidate is going to achieve a good mark, all the steps in the argument must be clearly argued. In particular, it is not actually given in the question that there are four of each colour in the grid, but it is an easy deduction – by considering the corner sub-squares.

The mark-scheme was quite subtle and this proved to be a difficult question to mark. Proving that two adjacent corners had to be coloured differently was worth 3 marks (if clearly argued), so it was necessary to tackle the opposite corners to progress to the 10- region. The construction argument was more difficult to assess since there are so many different approaches, and depends whether you begin or end with the central sub-square.

There were some attempts at a more traditional counting argument, but these were not convincing.

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Q6

This was a very difficult question and the markers were impressed by those candidates who produced complete solutions.

There was clearly some confusion about what was required. Many candidates tried specific values for m and n and then spent time considering equations such as $x^3 = y + z$ (for $m = n = 1$) or $x^7 = y^2 + z^3$ (for $m = 2$ and $n = 3$). Such efforts were not worth any credit. It is essential that both m and n are allowed to have any positive values.

Having said this, it seems unlikely that, for arbitrary m and n, we are going to have much luck with equations such as $3^m + 5^n = 2^{mn+1}$, whereas $2^m + 2^n = 2^{mn+1}$ looks at least plausible, so long as there is something extra to allow us to add the first two terms. This suggests taking $y = 2^n$ and $z = 2^m$, so $\left(2^m\right)^n + \left(2^n\right)^m = 2{\times}2^{mn} = 2^{mn+1}$, meaning that $x = 2$. This insight was worth 3 marks.

Making further progress was very difficult. It was necessary to produce an infinite set of triples (x, y, z) which 'worked' in the same way.

Of the few who did get this far there were various ways to go and as many different ways were seen as had been considered for marking. The general approach is to generate new solutions from old ones or adjust one factor which can take infinitely many values. Adjusting the powers to fit the same base or adjusting the base for the same powers both led to solutions.

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