UK Maths Trust

Mathematical Olympiad for Girls

Wednesday 25 September 2024

Organised by the United Kingdom Mathematics Trust

supported by Jane Street Overleaf

Instructions

1. Do not turn over the page until told to do so.
2. Time allowed: $2\frac{1}{2}$ hours.
3. Each question carries 10 marks.
4. Questions 1 and 4 require no explanations in words, but you should carefully follow the instructions on each answer sheet. Partial marks may be awarded for some correct values in tables even if the final answers are incorrect.
5. Questions 2, 3 and 5 require full written explanations. If your solution involves calculations, equations, tables, etc., explain where these come from and how you are using them. Explain how the steps of your solution link together, and give full proofs of assertions that you make. Answers alone will gain few marks (if any).
6. Partial marks may be awarded for good ideas, so try to hand in everything that documents your thinking on the problem — the more clearly written the better.
   However, one complete solution will gain more credit than several unfinished attempts.
7. Earlier questions tend to be easier. Questions have multiple parts. Often earlier parts introduce results or ideas useful in solving later parts of the problem.
8. The use of rulers and compasses is allowed, but calculators and protractors are forbidden.
9. You may use rough paper to note down your ideas, but you should write up your solution on the answer sheet provided for each question.
10. Start each question on an official master answer sheet that has a QR code on it.
    You may use additional sheets (blank or lined paper only). On each additional sheet please write the number of the question in the top left-hand corner, followed by the QR code digits following the ':' symbol. Please do not write your name or initials on additional sheets.
11. Write on one side of the paper only.
12. Arrange your answer sheets in question order before they are collected. Please remove blank answer sheets and rough paper.
13. To accommodate candidates sitting at other times, please do not discuss the paper on the internet until 08:00 BST on Friday 27 September, when the solutions video will be released at ukmt.org.uk/competition-papers. Candidates in time zones more than 5 hours ahead of GMT must sit the paper on Thursday 26 September (as defined locally).

Enquiries about the Mathematical Olympiad for Girls should be sent to:

challenges@ukmt.org.uk

www.ukmt.org.uk

1. For this question, follow the instructions on the answer sheet.

ABC is a right-angled triangle, with right-angle at B and side lengths AB and BC a whole number of centimetres.

F lies on AB, D lies on BC and E lies on AC, as shown in Figure 1 below, so that $BDEF$ is a rectangle.

Denote lengths, in centimetres, $BF = a$, $AF = x$, $BD = c$ and $CD = y$.

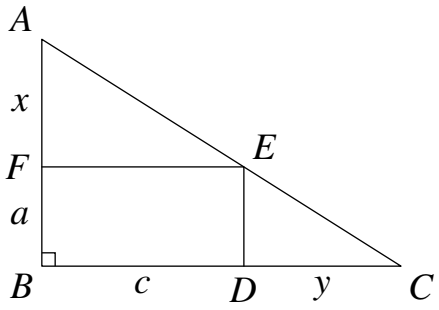

Figure 1

(a) Given that $a = 2$, $c = 6$ and $y = 3$, find x. (2 marks)

(b) It is now given that the area of rectangle $BDEF$ is 9 cm^2 and its dimensions are a whole number of centimetres.

Find all possible values for the length BC in centimetres. (8 marks)

2. This question requires full written explanations.

In Figure 2 below, each of the five circles A, B, C, D, E is to be coloured so that adjacent circles are different colours.

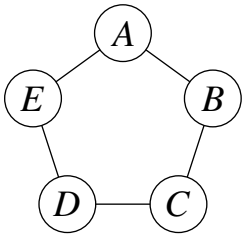

Figure 2

(a) In this part, circle A is to be coloured red and circle B yellow.

(i) Copy the diagram and show one possible way to colour the circles using colours red, yellow and green. Write 'R', 'Y' or 'G' in each circle to indicate its colour.

(ii) Explain why it is not possible to colour all five circles using only red and yellow. You should use labels A, B, C, D, E to refer to the circles.

(2 marks)

(b) In this part, the colours to be used are purple, orange and white. How many different ways are there to colour the circles using these three colours, so that adjacent circles are different colours? (8 marks)

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3. This question requires full written explanations.

(a) Figure 3 shows a semi-circle with centre O and diameter PQ and a point R on the semi-circle. By expressing other angles in the diagram in terms of p and q, prove that the angle in a semi-circle is 90°. You must clearly state any geometrical facts you use.

(2 marks)

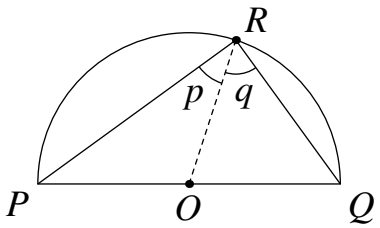

Figure 3

(b) Figure 4 shows an isosceles triangle ABC, with $AB = AC$. A semi-circle with centre M and diameter AC intersects AB at N.

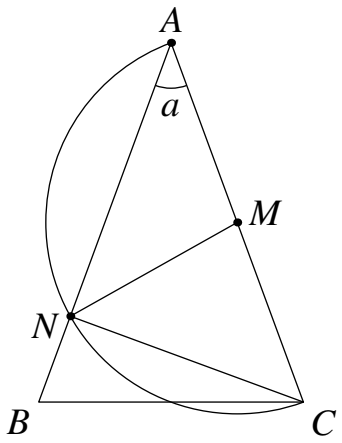

Figure 4

Given that $\angle AMN = 6\angle BCN$, find the size of angle $\angle BAC$, labelled a.

You must clearly state any geometrical facts you use. (8 marks)

4. For this question, follow the instructions on the answer sheet.

(a) The equation $uv + 3u - 3v - 18 = 0$ is equivalent to $(u - a)(v + b) = 9$. Write down the values of a and b. (1 mark)

(b) Hence find all integer solutions of the equation $uv + 3u - 3v - 18 = 0$.

(3 marks)

(Note: 'integers' means whole numbers, which can be positive, negative or zero.)

(c) Find all integer solutions of the equation $x^2 - y^2 + 6y - 18 = 0$.

(6 marks)

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5. This question requires full written explanations.

The positive integers from 1 to N are equally spaced around a circle in such a way that the sum of any two neighbouring numbers is equal to the sum of the two numbers diametrically opposite them.

(a) Figure 5 shows an example with $N = 6$. Number 1 is diametrically opposite number 4 and number 2 is diametrically opposite a.

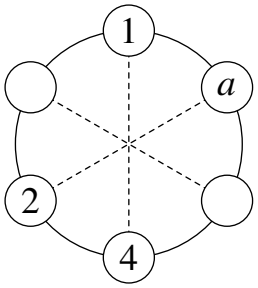

Figure 5

Copy and complete the diagram to show one possible way of arranging numbers 1 to 6. No explanation is needed in this part. (1 mark)

In the rest of this question, $N = 222$.

(b) Show that the difference between any pair of diametrically opposite numbers is the same. You may find it helpful to refer to Figure 6 in your explanation. (1 mark)

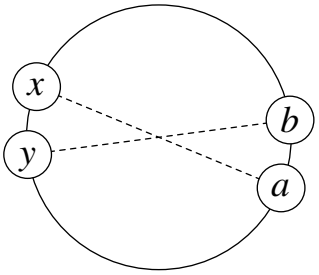

Figure 6

(c) Suppose that there is an arrangement in which number 41 is opposite number 1. By considering which numbers would need to be placed opposite 40, 120 and 200, show that such an arrangement is not possible. (1 mark)

(d) Find all possible numbers which could be placed opposite number 1. You need to explain why those are the only possibilities. In each case, you also need to explain why it is possible to arrange the remaining numbers. (7 marks)

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