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Andrew Jobbings Senior Kangaroo

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Solutions

These are polished solutions and do not illustrate the process of failed ideas and rough work by which candidates may arrive at their own solutions.

It is not intended that these solutions should be thought of as the 'best' possible solutions and the ideas of readers may be equally meritorious.

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1. The seventh day of an unknown month in an unknown year is a Tuesday. The nth day of the following month is the first Tuesday of that month. What is the product of all the possible values of n?

SOLUTION **840**

The seventh is a Tuesday, so the twenty-eighth will also be a Tuesday. For a 28, 29, 30 or 31 day month, the first Tuesday of the next month will then fall on the 7th, 6th, 5th or 4th respectively. The product of these dates is $7 \times 6 \times 5 \times 4 = 840$.

2. Each of the following expressions has a different value. Which value is greatest?

$$2 - 0 \times 2 + 4 + 11 \times 20$$
$$2 + 0 \times 2 \times 4 + 11 \times 20$$
$$2 \times 0 + 2 \times 4 + 11 \times 20$$
$$2 \times (0 + 2 + 4) + 11 \times 20$$

SOLUTION **232**

The values are:
$2 - 0 \times 2 + 4 + 11 \times 20 = 2 - 0 + 4 + 220 = 226$
$2 + 0 \times 2 \times 4 + 11 \times 20 = 2 + 0 + 220 = 222$
$2 \times 0 + 2 \times 4 + 11 \times 20 = 0 + 8 + 220 = 228$
$2 \times (0 + 2 + 4) + 11 \times 20 = 12 + 220 = 232$

3. Kevin begins with eleven stones and a mallet. Every time he hits a stone with the mallet it breaks into exactly eight pieces. Kevin keeps hitting stones with his mallet, but stops before he has 1000 stones. What is the largest number of stones he could end up with?

SOLUTION **998**

Every time Kevin hits a stone he destroys one stone and creates eight new ones, that is he increases his number of stones by $8 - 1 = 7$. He starts with 11 stones, which has a remainder of four when divided by 7. Repeatedly adding 7 will not change this remainder, so we find the largest three-digit number with this remainder. Since $994 = 7 \times 142$, the answer is 998.

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4. The number 35 can be written as the sum of four distinct primes in two different ways: $35 = 2 + 3 + 13 + 17$ and $35 = 2 + 3 + 11 + 19$.
What is the *smallest* number that can be written as the sum of four distinct primes in two different ways?

SOLUTION **023**

The sum of the smallest four odd primes is $3 + 5 + 7 + 11 = 26$. So any such sum that is less than 26 must include 2.
If 2 is to be included in the sum, then the sum must be an odd number.
The smallest four primes are 2, 3, 5, and 7, and $2 + 3 + 5 + 7 = 17$, which is the only way to write 17.
The next possible such sum (using the three smallest primes and the fifth smallest prime) is $2 + 3 + 5 + 11 = 21$, and this is the only way to write 21 as such a sum.
We now consider 23. Because $2 + 3 + 7 + 11 = 2 + 3 + 5 + 13 = 23$ we see that 23 is the smallest such sum.

5. What is the value of $|\sqrt{123} - 456| + |\sqrt{123} + 456|$?
The modulus function, $|x|$ is given by $|x| = x$ for $x \geq 0$, and $|x| = -x$ for $x < 0$.

SOLUTION **912**

We observe that $\sqrt{123} < \sqrt{144} = 12 < 456$. Therefore, $|\sqrt{123} - 456| = 456 - \sqrt{123}$.
The value of $|\sqrt{123} - 456| + |\sqrt{123} + 456|$ is $456 - \sqrt{123} + \sqrt{123} + 456 = 456 + 456 = 912$.

6. The equation $x^2 - 70x = 2024$ has solutions a, b with $b > a$. What is the value of $10b - a^2$?

SOLUTION **436**

We can solve this equation by completing the square: $x^2 - 70x = (x - 35)^2 - 1225 = 2024$.
Therefore $(x - 35)^2 = 3249$ and $x - 35 = \pm 57$, that is $x = 22$ or $x = 92$.
We have $b = 92$, $a = 22$, and $10b - a^2 = 920 - 484 = 436$.

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7. The triangle ABC is such that $AB = 7$, $BC = 3$, and $CA = 5$.
Furthermore, we are given that $\angle ACB = 120°$.
The area of triangle ABC is M. What is the value of M^2, correct to the nearest integer?

SOLUTION **042**

There are a multitude of different methods for solving this problem including Heron's formula, or use of the $\frac{1}{2}ab\sin C$ formula, for those students who are aware of them.
Another approach is to observe the exterior angle at C is 60^o and complete an equilateral triangle ACD, where D lies on CB projected. The perpendicular from A to CD extended has length $\sqrt{5^2 - (2.5)^2} = \sqrt{18.75}$. Since this altitude is also perpendicular to BC, the required area is $\frac{1}{2} \times 3 \times \sqrt{18.75} = \frac{1}{2} \times 3 \times \sqrt{6.25}\sqrt{3} = \frac{1}{2} \times 3 \times\ 2.5 \times \sqrt{3} = \frac{15\sqrt{3}}{4}$.
Therefore, $M^2 = \frac{15^2 \times 3}{4^2} = \frac{675}{16} = 42 + \frac{3}{16}$. This is 42 correct to the nearest integer.

8. At a football match between Glasgow Belles and Doncaster City, five supporters each made a prediction:

1. The game will end in a draw;
2. Glasgow will score;
3. Glasgow will win;
4. Glasgow will not lose;
5. Exactly five goals will be scored.

What is the maximum number of these predictions that could simultaneously come true?

SOLUTION **004**

We observe that predictions (1) and (5) cannot simultaneously occur. The scenario where Glasgow Belles win 4-1 satisfies predictions (2), (3), (4) and (5) and therefore the maximum number that could simultaneously happen is four.

9. The number $N = 24^{2024} + 24^{2025}$ has some factors between 1 and 25.
What is the sum of all those integers between 1 and 25 which are *not* factors of N?

SOLUTION **147**

We re-write $N = 24^{2024} + 24^{2025}$ as $N = 24^{2024} \times (1 + 24) = 24^{2024} \times 25 = 2^{3\times 2024} \times 3^{2024} \times 5^2$.
Any integer between 1 and 25 which can be expressed as a product of powers of 2, 3, and 5 (for example, $20 = 2^2 \times 5$), is a factor of N.
There are only nine integers which cannot be expressed in such a way.
These integers have sum $7 + 11 + 13 + 14 + 17 + 19 + 21 + 22 + 23 = 147$.

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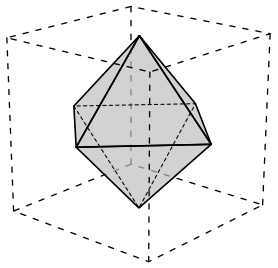
10. An octahedron is inscribed in a cube of side-length 12. The vertices of the octahedron are at the centre of the faces of the cube.
The surface area of the octahedron is $a \times \sqrt{b}$, where b is a prime number. What is the value of $a + 10b$?



SOLUTION **174**

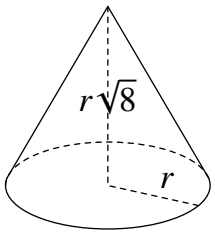
By Pythagoras, the distance between the midpoints of two adjacent faces of the cube is $\sqrt{6^2 + 6^2} = 6 \times \sqrt{2}$.
Each face of the octahedron is an equilateral triangle with area
$\frac{1}{2} \times 6\sqrt{2} \times 6\sqrt{2} \times \sin 60 = \frac{1}{2} \times 6 \times 6 \times 2 \times \frac{\sqrt{3}}{2} = 18\sqrt{3}$.
Therefore, the octahedron has surface area $8 \times 18\sqrt{3} = 144\sqrt{3}$.
This is in the form $a + 10b$ with $a = 144$, $b = 3$, and $a + 10b = 144 + 30 = 174$.

11. The number $L = \frac{24!}{20!}$ is one less than a square, S^2.
What is the value of S?

SOLUTION **505**

We observe that $L = \frac{24!}{20!} = 24 \times 23 \times 22 \times 21 = (24 \times 21) \times (23 \times 22) = 504 \times 506 = (505 - 1) \times (505 + 1) = 505^2 - 1$.

12. A solid right circular cone whose base has radius r metres has total surface area (including the circular base) of A metres squared and a volume of V metres cubed.
The perpendicular height of the cone is $r\sqrt{8}$ metres.
Given that $\frac{V}{A} = \sqrt{5000}$, what is the value of r?



SOLUTION **300**

The total surface area of the cone is given by the formula $\pi rl + \pi r^2$, where l is the slant height of the cone.
By Pythagoras, $l = \sqrt{8r^2 + r^2} = 3r$.
We have $A = \pi \times r \times 3r + \pi \times r^2 = 4\pi r^2$.
The volume of the cone is given by the formula $\frac{1}{3}\pi r^2 h$, where h is the perpendicular height of the cone.
We have $V = \frac{1}{3}\pi r^2 \times r\sqrt{8} = \frac{\pi r^3 \sqrt{8}}{3}$.
Since $\frac{V}{A} = \sqrt{5000}$, $\frac{\pi r^3 \sqrt{8}}{3} \times \frac{1}{4\pi r^2} = \sqrt{5000}$.
Therefore, $\frac{r\sqrt{8}}{12} = \sqrt{5000}$. Hence, $r = 12 \times \frac{\sqrt{5000}}{\sqrt{8}} = 12 \times \sqrt{\frac{5000}{8}} = 12 \times \sqrt{625} = 12 \times 25 = 300$.

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13. Wil goes on a run. Her pedometer is unfortunately rather basic, and records a distance of 40 km, to the nearest integer, and an average speed of 10 km/h, to the nearest integer. Wil works out her greatest possible and least possible times, and expresses the difference of these times as a fraction in its lowest terms, $\frac{a}{b}$.
What is the value of $a + 2b$?

SOLUTION **998**

Her greatest and least times are $\frac{40.5}{9.5}$ and $\frac{39.5}{10.5}$ respectively. These have a difference of $\frac{405}{95} - \frac{395}{105} = \frac{81}{19} - \frac{79}{21} = \frac{81\times21-79\times19}{19\times21} = \frac{1701-1501}{399} = \frac{200}{399}$.
Therefore, $a = 200$, $b = 399$, and $a + 2b = 998$.

14. The function f has the property that $f(x + y) = f(x) \times f(y)$ for all values of x and y. Given that $f(x) > 0$ for all values of x, and also that $f(24) = 4096$, what is $f(18)$?

SOLUTION **512**

$4096 = f(24) = f(12 + 12) = f(12) \times f(12)$, so $f(12) = \sqrt{4096} = 64$.
Therefore, $64 = f(12) = f(6 + 6) = f(6) \times f(6)$, so $f(6) = \sqrt{64} = 8$.
Hence, $f(18) = f(12 + 6) = f(12) \times f(6) = 64 \times 8 = 512$.

15. Four friends named Ada, Brin, Chris and Debbie are each carrying a different amount of money. These amounts are £111, £222, £333 and £444.

Ada says: I am neither carrying the most nor the least.

Brin says: I am not carrying the least.

Chris says: I am carrying the most.

Debbie says: I am carrying the least.

Exactly one of the friends is lying. How many letters does the lying friend have in her/his name?

SOLUTION **005**

If Ada is lying then either Chris or Debbie must also be lying, meaning more than one friend is lying. So Ada must be telling the truth.
If Brin is lying then Debbie is also lying, so Brin must be telling the truth.
If Debbie is lying then none of Debbie, Brin or Ada are carrying the least, so Chris would be carrying the least. This would mean Chris would also be lying, so Debbie must be telling the truth.
So Chris must be lying and Chris is not carrying the most. This is consistent with Brin carrying most, then Chris and Ada in some order, then Debbie.
Chris has five letters in his name, so the answer is 005.

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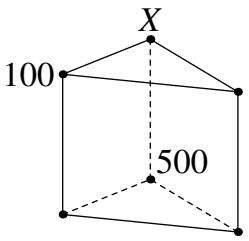
16. The prism in the picture is formed of two triangles and three squares. The six vertices are assigned labels 100, 200, 300, 400, 500, and 600 in such a way that the sum of the four labels of the vertices of each square is the same for all three squares. Labels 100 and 500 are shown.
What number is at the vertex labelled X?



SOLUTION **200**

For the front square, label the vertices 100, a, b and c, in anticlockwise order. Then $100 + a + b + c = X + 500 + b + c$. Therefore, $a - X = 400$ from which it follows at once that $a = 600$ and $X = 200$ (with b and c being 300 and 400 in some order).

17. The parabola $y = x^2 - 10x + 24$ has two distinct x-intercepts and a y-intercept.
The circle which goes through these three intercepts meets the parabola at a fourth point, P.
What is the distance of P from the origin?

SOLUTION **026**

By solving the equation $0 = x^2 - 10x + 24$, we determine that x-intercepts are at 4 and 6, while the y-intercept is at 24.
The parabola has a line of symmetry equidistant from its roots, i.e. with equation $x = 5$. Since $(4, 0)$ and $(6, 0)$ are at opposite ends of a chord to the circle, the line of symmetry $x = 5$ is a diameter of the circle. We may therefore obtain the coordinates of P by reflecting the parabola's y-intercept in the line $x = 5$.
So P is at $(10, 24)$ and is a distance $\sqrt{10^2 + 24^2} = 26$ from the origin.

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18. Zarah has a bag containing exactly four blue counters and exactly four red counters, and a large supply, outside the bag, of extra blue and red counters.
She may, on any given *move*, do one of four different things:

- Remove a blue counter from the bag;
- Remove a red counter from the bag;
- Add a blue counter to the bag;
- Add a red counter to the bag.

At each move, she selects one of those four options at random and with equal probability.
The probability that after four consecutive moves, Zarah's bag again contains exactly four blue counters and exactly four red counters is $\frac{V}{W}$ when written in its lowest terms.
What is the value of $10V + W$?

SOLUTION **154**

There is a one in four chance that Zarah's second move is the opposite of her first. In this case, Zarah will need her fourth move to be the opposite of her third, which also has a one in four chance.
There is a one in four chance that Zarah's second move is identical to her first. In this case, both her third and fourth moves will need to be the exact opposite to her first and second to return to the original state, with a one in four chance for each of the third and fourth moves being such.
There is a two in four chance that Zarah's second move is neither opposite nor identical to her first (i.e. the first and second moves involve different coloured counters). In this case there is a one in two chance that her third move will be opposite to either of her first or her second moves, followed by a one in four chance that her fourth move is opposite to whichever of her first or second moves was not *undone* during move three.
The respective probabilities of these three cases are $\frac{1}{4}\times\frac{1}{4} = \frac{1}{16}$, $\frac{1}{4}\times\frac{1}{4}\times\frac{1}{4} = \frac{1}{64}$, and $\frac{2}{4}\times\frac{1}{2}\times\frac{1}{4} = \frac{1}{16}$, with sum $\frac{1}{16} + \frac{1}{16} + \frac{1}{64} = \frac{9}{64}$.
Therefore, $V = 9$, $W = 64$, and $10V + W = 90 + 64 = 154$.

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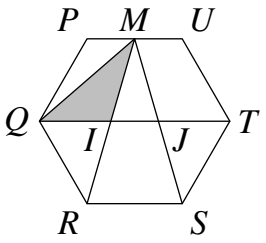
19. A regular hexagon $PQRSTU$ is shown. The midpoint of UP is M. Point I is the intersection between line segments RM and QT, and point J is the intersection between line segments SM and QT.
The area of triangle RMS is 920. What is the area of triangle QMI?



SOLUTION **345**

Let the side-length of the hexagon be $2x$.
The perpendicular height of triangle RMS is $\frac{2\times920}{2x} = \frac{920}{x}$.
By symmetry, the perpendicular height of triangle QMI is $\frac{1}{2} \times \frac{920}{x} = \frac{460}{x}$.
Triangles MIJ and MRS are similar, and IJ therefore has length $\frac{1}{2} \times 2x = x$.
The diagonals PS, QT and RU would divide the original hexagon into six congruent equilateral triangles, from which it is easy to conclude that $QT = 2x + 2x = 4x$.
Therefore, $QI + JT = QT - IJ = 4x - x = 3x$, and by symmetry, $QI = JT = \frac{3x}{2}$.
So the area of triangle QMI is $\frac{1}{2} \times \frac{3x}{2} \times \frac{460}{x} = 345$.

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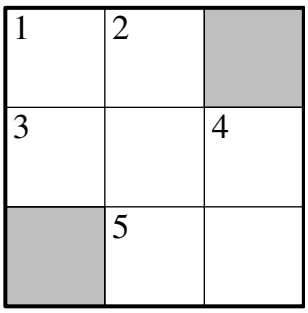
20. Each cell in this cross-number can be filled with a non-zero digit so that all of the conditions in the clues are satisfied. The digits used are not necessarily distinct.



ACROSS
1. A prime.
3. The answer to this Kangaroo question.
5. Shares a two-digit common factor with 1 DOWN.

DOWN
1. The sum of two different sixth powers.
2. 101 less than a square number.
4. The sum of all the digits in this crossnumber, once correctly completed.

SOLUTION **594**

The only possibility for 1 DOWN that is is a two-digit number is 65.
We then know that 1 ACROSS is either 61 or 67.
If 1 ACROSS is 61 then 2 DOWN is 124, 155, or 188.
We now consider 5 ACROSS. The only two-digit factors of 65 (1 DOWN) are 13 and 65.
So 2 DOWN cannot be 124 since there are no multiples of 13 or 65 between 41 and 49.
Also, 2 DOWN cannot be 188 since there are no multiples of 13 or 65 between 81 and 89.
If 2 DOWN were 155 then 5 ACROSS would be 52 (the only multiple of 13 or 65 between 51 and 59).
At this point the *known* digits in the completed crossnumber would sum to 24. The maximum digit-sum for the completed crossnumber is therefore 24 + 9 = 33, meaning 8 is the only case to check (since we know 4 DOWN must end with a 2). However, this yields a contradiction since the overall digit-sum is not 82.
So 2 DOWN cannot be 155, so 1 ACROSS cannot be 61 and must be 67.
2 DOWN must be 799 (observe that while 740 is also 101 less than a square and has the correct hundreds digit, we are told in the question that all cells are filled with *non-zero* digits).
5 ACROSS is then 91 (the only multiple of 13 or 65 between 91 and 99).
At this point the *known* digits in the completed crossnumber would sum to 37. The maximum digit-sum for the completed crossnumber is therefore 37 + 9 = 46, and since we already know that in this case 4 DOWN ends with a 1 we now merely need to verify that if 4 DOWN = 41 that all the clues are satisfied.
The digit-sum in this case is indeed 41 and the answer to 3 ACROSS and this question is 594.

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