UK Maths Trust

Intermediate Mathematical Olympiad

Cayley paper

Thursday 20 March 2025

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England & Wales: Year 9 or below | *Scotland: S2 or below* | *Northern Ireland: Year 10 or below*

These problems are meant to be challenging.

Try to finish whole questions even if you cannot do many; you will have done well if you hand in a complete solution to two or more questions.

Instructions

1. Time allowed: **2 hours**.
2. **Full written solutions – not just answers – are required**, with complete proofs of any assertions you may make. Marks awarded will depend on the clarity of your mathematical presentation. Work in rough first, and then write up your best attempt. Do not hand in rough work.
3. **One complete solution will gain more credit than several unfinished attempts**.
4. **Each question carries 10 marks**.
5. The use of rulers, set squares and compasses is allowed, but **calculators and protractors are forbidden**.
6. Start each question on an official answer sheet on which there is a **QR code**.
7. If you use additional sheets of (plain or lined) paper for a question, please write the following in the top left-hand corner of each sheet. (i) The question number. (ii) The page number for that question. (iii) The digits following the ':' from the question's answer sheet QR code. **Please do not write your name or initials on additional sheets**.
8. **Write on one side of the paper only**. Make sure your writing and diagrams are clear and not too faint. (Your work will be scanned for marking.)
9. **Arrange your answer sheets in question order before they are collected**. If you are not submitting work for a particular problem, please remove the associated answer sheet.
10. To accommodate candidates in other time zones, please do not discuss the paper online until **12pm GMT on Saturday 22 March**, when the video solutions will be released.
11. Do not turn over until told to do so.

Enquiries about the Intermediate Mathematical Olympiad should be sent to:

challenges@ukmt.org.uk *www.ukmt.org.uk*

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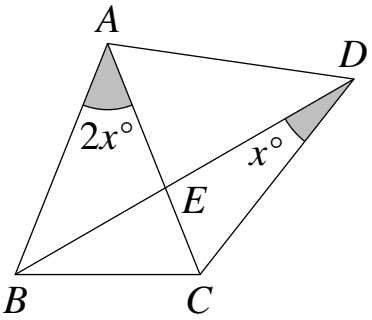
Advice to candidates

- ◇ *Earlier questions tend to be easier. In general, you are advised to concentrate on these problems first. It is more important to complete a small number of questions than to try all the problems.*
- ◇ *Spend time working carefully on one question before attempting another.*
- ◇ *Numerical answers should be fully simplified and exact. They may contain symbols such as π, fractions or square roots, but not decimal approximations.*
- ◇ *Stating answers, even correct answers, without justification will earn you very few marks.*
- ◇ *You are encouraged to use geometrical instruments to construct large, accurate diagrams for geometry problems.*

1. A number N is a two-digit positive integer. It can be written as $\overline{ab}$ to show its digits a and b. How can N written in terms of a and b in standard algebra? Find all possible N with digits a and b, where $N = \overline{ab}$ and $N = a + b^2$.

2. In the quadrilateral $ABCD$, $AB = AD$ and $AC = DC$. The diagonals of the quadrilateral meet at E and $AE = EB$.
Also, $\angle BAC = 2\angle BDC$ as shown.
Find the value of $\angle BDC$.
Hence prove that triangle ABC is isosceles.



3. A cuboid has dimensions 7 by 8 by x where $x > 8$. A water tank is a cylinder with base area, a. The cuboid stays at the bottom when water is added.
When the cuboid has the 8 by x face down, a volume, v, of water is added to the tank to a depth of 7, level with the top of the cuboid.
When the cuboid has the 7 by x face down, the water level drops by 1.
When the cuboid has the 7 by 8 face down, the water level again drops by 1.
Find how much the water level drops when the cuboid is removed.

4. A tile is one of the two triangles made by cutting a square of area 3 in half along the diagonal. In attempting to tile a 49 by 49 square floor using these tiles with none of them overlapping and none going outside the square, it is impossible to tile it fully. Find the maximum number of these tiles which can be used in this tiling.
Demonstrate how to do this and prove that no more tiles will fit.

5. A keypad is a 3 by 3 grid with the digits 1 to 3 in order on the top row, 4 to 6 in order on the second row and 7 to 9 in order on the third row.
Seb has the keypad upside-down (i.e rotated through 180°) and types his three-digit PIN as if the keypad were the right way up. The number he types in turns out to be k times his original PIN where k is a positive integer. Find all possible PINs and prove there are no others.

6. 2025 penguins (675 each of emperors, kings and gentoos) stand in a line from left to right. They may move by performing the penguin shuffle in which an emperor takes a king's place, a gentoo takes the position vacated by the emperor and the displaced king takes the place vacated by the gentoo. Prove that, no matter what the starting order, it is possible to place the emperors on the left, kings in the middle and gentoos on the right after a finite number of shuffles.

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