UK Maths Trust

Intermediate Mathematical Olympiad

Maclaurin paper

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Solutions

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1. A cuboid box has height 10 and a square base of side length x. It is put in a cylindrical water tank with base a circle of area 360. The cuboid stays at the bottom when water is added.
When the box is in the tank of water with a square face against the bottom of the tank, the water reaches the top of the box.
When the box is in the tank of water with a non-square face against the bottom of the tank, the water level drops by 1.
The box is removed from the tank. How much further does the water level drop then?

SOLUTION

Let v be the volume of water.

The first scenario gives this equation.

$$v = 10(360 - x^2)$$

The second scenario gives this equation.

$$v = 9(360 - 10x)$$

Equating the two expressions for v gives this equation.

$$10(360 - x^2) = 9(360 - 10x)$$

This rearranges to $x^2 - 9x - 36 = 0$.

So, $(x - 12)(x + 3) = 0$ and $x = 12$ or $x = -3$.

Since $x > 0$, $x = 12$.

Substituting in for x gives $v = 2160$.

When the cuboid is removed, the height of the water level is $\frac{v}{360} = \frac{2160}{360} = 6$.

Therefore, the water drops from 9 to 6, which is a further drop of 3.

2. Alice and Charlie play a game, taking it in turns with Alice going first.
On a blackboard is written a three-digit number. If the current number on the blackboard is n, a move consists of choosing a non-zero digit, k, of n and replacing n with $n - k$ on the blackboard. This is repeated until the number 100 is written on the blackboard. The player who writes the number 100 wins.
(a) If the starting number is 125, Alice can always win. State Alice's first move and how Alice responds to whatever move Charlie makes at each stage.
(b) Find, with proof, for which starting values Charlie has a winning strategy.

SOLUTION

(a) Alice should subtract 5 and write 120.

Charlie can subtract 1 and write 119 or subtract 2 and write 118.

Alice can subtract 9 from 119 or 8 from 118 and in both cases write 110.

Charlie can only subtract 1 and write 109.

Alice can then subtract 9 and write 100 and win the game.

(b) Charlie can win whenever the starting number ends in 0.

If the starting number does not end in 0, Alice can subtract the units digit and write a number ending in 0. Whatever Charlie does, he will subtract a number less than 10 and will write down a number not ending in 0. Alice is then in a position to repeat the process.

By this strategy, Alice writes down only numbers ending in 0 and Charlie never writes numbers ending in 0, so Alice will write down 100 and win.

However, if the starting number ends in zero, Alice must write down a number not ending in 0. Charlie can then use the strategy used by Alice above to win the game.

Therefore Charlie can win whenever the starting number ends in 0 and cannot win with any other starting numbers.

3. The digits of 2024 have the property that $7^2 + 7^0 + 7^2 + 7^4 = 2500$ ends in two zeroes. Find how many sets of solutions there are for single digits a, b, c and d so that $7^a + 7^b + 7^c + 7^d$ ends in two zeroes.
Note that $(a, b, c, d) = (2, 0, 2, 4)$ and $(a, b, c, d) = (0, 2, 2, 4)$ would be two different sets of solutions.

SOLUTION

Because 7^4 ends in 01, the last two digits of powers of 7 will repeat 01, 07, 49, 43.

When the last two digits are 01, the digit in the power of 7 could be 0, 4 or 8.

When the last two digits are 07, the digit in the power of 7 could be 1, 5 or 9.

When the last two digits are 49, the digit in the power of 7 could be 2 or 6.

When the last two digits are 43, the digit in the power of 7 could be 3 or 7.

The only possible sums of these numbers ending in two zeroes are 01+01+49+49, 01+07+43+49 and 07 + 07 + 43 + 43.

In the first and third cases, there are 6 ways to order the four numbers in the sum.

In the first case, there are three options for the digits leading to each of the 01's and two options for the digits leading to each of the 49's. This leads to $6 \times 3 \times 3 \times 2 \times 2 = 216$ solutions.

In the third case, there are three options for the digits leading to each of the 07's and two options for the digits leading to each of the 43's. This leads to $6 \times 3 \times 3 \times 2 \times 2 = 216$ solutions.

In the second case, there are 24 ways to order the four numbers in the sum, three options for the digits leading to each of 01 and 07 and two options for the digits leading to each of 43 and 49. This leads to $24 \times 3 \times 3 \times 2 \times 2 = 864$ solutions.

In total there are $216 + 216 + 864 = 1296$ solutions.

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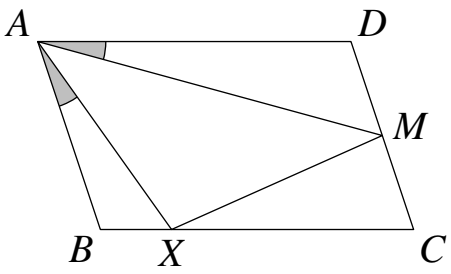
4. In a parallelogram $ABCD$, M is the midpoint of DC. There is a point X on BC such that $\angle MAX = 30°$, $\angle BAX = \angle MAD$ and $AX = XM$. Find the ratio of $BX : XC$.



SOLUTION

$\angle ABX = \angle ADM$ because opposite angles in parallelograms are equal.

Therefore $\triangle ABX \sim \triangle ADM$ because they have two, and therefore three, angles equal.

If Y is the midpoint of AM, then $AY = \frac{\sqrt{3}}{2}AX$, so $AM = \sqrt{3}AX$.

The scale factor between the triangles is $\sqrt{3}$.

Therefore, $AD = \sqrt{3}AB = \sqrt{3}DC = 2\sqrt{3}MD = 2\sqrt{3}(\sqrt{3}BX) = 6BX$.

Therefore, $CX = BC - XB = AD - XB = 6XB - XB = 5XB$.

So, $\frac{CX}{XB} = 5$ and $BX : XC = 1 : 5$.

5. Let m and n be positive integers such that m divides $n + 20$ and n divides $m + 25$. What is the maximum possible value of $m + n$?

SOLUTION

If m divides $n + 20$, then $n + 20 = pm$ for some positive integer p.

If n divides $m + 25$, then $m + 25 = qn$ for some positive integer q.

Adding these equations gives $m + n + 45 = pm + qn$.

Rearranging and factorising gives $45 = (p - 1)m + (q - 1)n$.

If $p = 1$, $m = n + 20$, so m is maximised when n is maximised.

This happens when $q - 1$ is the smallest non-zero value, so $q = 2$.

In this case, $n = 45$, $m = 65$ and $m + n = 110$.

If $q = 1$, $n = m + 25$, so n is maximised when m is maximised.

This happens when $p - 1$ is the smallest non-zero value, so $p = 2$.

In this case, $m = 45$, $n = 70$ and $m + n = 115$.

If $p, q \geq 2$, $m + n \leq (p - 1)m + (q - 1)n = 45$, so cannot be larger than 115.

Note that it is not possible for $p, q = 1$.

Therefore, the largest value of $m + n$ is 115.

6. There are fourteen rectangular tiles with sides of length 3 and 4 and one square tile with sides of length 1 to tile a square floor which is 13 by 13.
(a) Show that this is possible if the 3 by 4 tiles are cut into two 3-4-5 triangles.
(b) Prove that this is impossible without cutting any tiles.

SOLUTION

(a) This can be achieved with the following arrangement.

Use twelve of the rectangles (twenty-four triangles) to make four 4 by 9 rectangles. Put them on the outside of the square floor to leave a 5 by 5 square in the centre untiled.

Place the four remaining triangles with their hypotenuses on the sides of the 5 by 5 square, the side of length 3 starting at the corner on the clockwise side of the square and the side of length 4 starting at the other end of the side of the square.

The sides of adjacent tiles overlap because the angles at those corners sum to 90°.

This leaves a 1 by 1 square at the centre for the last tile.

(b) This is impossible because it would take at least 16 tiles to tile the grid.

All the tiles are rectangular, so the floor cannot be covered with the edges of the tiles not parallel to the edges of the floor.

Draw a 13 by 13 grid of squares on the floor.

Colour the top-left corner cell.

Colour every cell four to the right or four below a previously coloured cell, so 16 cells are coloured.

No tile can cover two of these coloured cells, so at least 16 tiles will be required to cover the floor.

There are only 15 tiles available, so it is impossible to cover the floor using these tiles without cutting them.