UK Maths Trust

British Mathematical Olympiad

Round 2

© 2026 UK Mathematics Trust

supported by Jane Street® Overleaf

Solutions

1. For any two positive integers m and n, we define $l(m,n)$ as their least common multiple and $h(m,n)$ as their highest common factor. Given a prime $p > 3$, let k denote the number of ordered pairs of positive integers (m,n) satisfying the equation

$$l(m,n) + h(m,n) = p^4.$$

Determine the smallest possible value of k across all choices of the prime $p > 3$.

Solution

For brevity, we write $h = h(m,n)$. Write $m = hM$ and $n = hN$, so M and N are coprime. We then have $l(m,n) = MNh$ and the equation becomes

$$h(MN+1) = MNh + h = p^4.$$

As p is prime, h must be a power of p. Since $MN + 1 \geq 2$, we cannot have $h = p^4$. We now consider the remaining four possibilities:

h	$MN+1$	MN	Fewest distinct prime factors of MN	Example when $p=5$
1	p^4	$p^4-1=(p-1)(p+1)(p^2+1)$	3	$624 = 2^4\cdot 3\cdot 13$
p	p^3	$p^3-1=(p-1)(p^2+p+1)$	2	$124 = 2^2\cdot 31$
p^2	p^2	$p^2-1=(p-1)(p+1)$	2	$24 = 2^3\cdot 3$
p^3	p	$p-1$	1	$4 = 2^2$

To justify the fourth column (for primes $p > 3$):

- $p - 1 \geq 2$, so $MN = p - 1$ has at least one prime factor.
- For $MN = p^2 - 1 = (p-1)(p+1)$: we have $\gcd(p-1, p+1) = 2$. If both $p-1$ and $p+1$ were powers of 2, then they would be consecutive powers of 2 differing by 2, forcing $p = 3$, which is not allowed. Hence an odd prime divides $p^2 - 1$, so $p^2 - 1$ has at least two distinct prime factors (namely 2 and some odd prime).
- For $MN = p^3 - 1 = (p-1)(p^2+p+1)$: $p-1$ is even, while p^2+p+1 is odd and greater than 1, so it has an odd prime factor. Thus there are at least two distinct prime factors.
- For $MN = p^4 - 1 = (p^2-1)(p^2+1)$: from above, $p^2 - 1$ has an odd prime factor. Also $p^2 + 1$ is 2 more than a multiple of 4 (as all odd squares leave remainder 1 when divided by 4), which means $p^2 + 1$ is even but not a power of 2; hence it has an odd prime factor. Any odd prime dividing both $p^2 - 1$ and $p^2 + 1$ would divide their difference 2, impossible, so these odd primes are different. Therefore $p^4 - 1$ has at least three distinct prime factors (namely 2 and at least two different odd primes).

Finally, once MN is fixed, to count coprime ordered pairs (M,N) with product MN we simply note each prime power dividing MN must go entirely into M or entirely into N (otherwise $\gcd(M,N) > 1$). So if MN has r distinct prime factors, there are exactly 2^r ordered coprime pairs (M,N).

Hence the smallest possible total number of solutions is

$$2^3 + 2^2 + 2^2 + 2^1 = 18.$$

© 2026 UK Mathematics Trust

This minimum is achieved when $p = 5$, as shown in the final column of the table. Therefore the smallest possible value of k is 18.

REMARK

For interest, the pairs with $m \geq n$ for $p = 5$ are given by:

$$(39, 16), (48, 13), (155, 20), (200, 75), (208, 3), (500, 125), (600, 25), (620, 5), (624, 1).$$

© 2026 UK Mathematics Trust

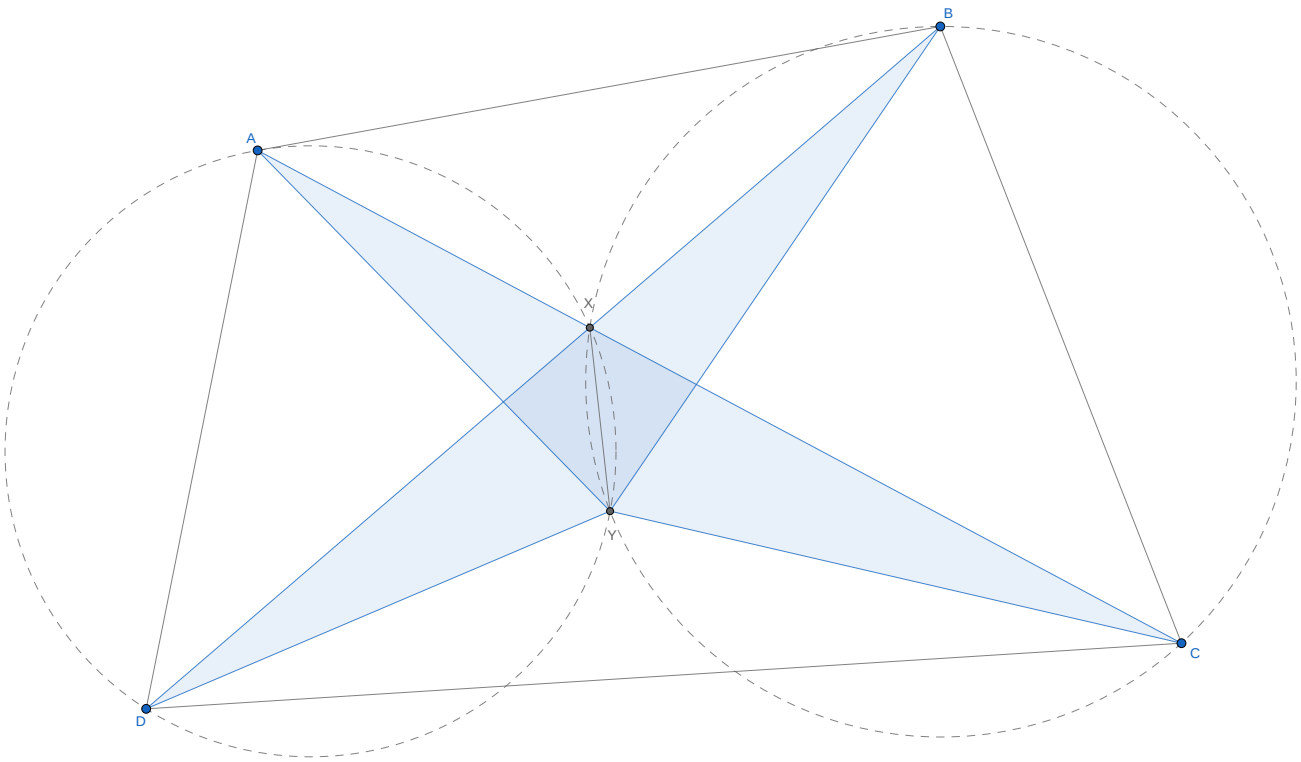
2. The convex quadrilateral $ABCD$ has sides AD and BC not parallel. The diagonals AC and BD intersect at X. The perpendicular bisectors of sides AD and BC meet at Y. Suppose that Y lies strictly inside triangle XCD.

Prove that $AC = BD$ if and only if XY bisects $\angle DXC$.

Solution

This is an *if and only if* problem so we need to prove implications in both directions. We do this by proving a chain of *if and only if* equivalences:

$$\begin{aligned} AC = BD &\iff \triangle AYC \text{ is congruent to } \triangle DYB && \text{(i)}\\ &\iff \angle YDX = \angle YAX \text{ and } \angle XBY = \angle XCY && \text{(ii)}\\ &\iff ADXY \text{ cyclic and } BCXY \text{ cyclic} && \text{(iii)}\\ &\iff XY \text{ bisects } \angle DXC. && \text{(iv)} \end{aligned}$$



We justify each of these in turn:

(i) Since Y lies on the perpendicular bisectors of both AD and BC, we have $AY = YD$ and $YC = YB$. Thus, $\triangle AYC$ and $\triangle DYB$ have two pairs of corresponding sides equal, so they are congruent if and only if their remaining sides are equal, i.e. if and only if $AC = BD$ (by SSS).

(ii) If the triangles are congruent, then corresponding angles are equal and using that X lies on both AC and BD we get

$$\angle YDX = \angle YDB = \angle YAC = \angle YAX.$$

Similarly, $\angle XBY = \angle XCY$.

Conversely, if both angle equalities hold then by considering the third angle in the triangles we have

$$\angle BYD = 180^\circ - \angle YDX - \angle XBY = 180^\circ - \angle YAX - \angle XCY = \angle CYA.$$

Together with $DY = AY$ and $BY = CY$, this gives $\triangle DYB \cong \triangle AYC$ by SAS.

(iii) Because A and D lie on the same side of XY, the equality $\angle YDX = \angle YAX$ is equivalent (by the angle in the same segment theorem and its converse) to $ADXY$ cyclic.

Similarly, because B and C lie on the same side of XY, the equality $\angle XBY = \angle XCY$ is equivalent to $BCXY$ cyclic.

(iv) If $ADXY$ is cyclic, then using $YA = YD$ and that equal arcs subtend equal angles we have

$$\angle DXY = \angle DAY = \angle YDA = 180^\circ - \angle AXY = \angle YXC$$

and so XY bisects $\angle DXC$.

For the converse, let the bisector of $\angle DXC$ intersect circle AXD at $Y' \neq X$. We are assuming that Y also lies on this line and we wish to show $Y = Y'$. As the perpendicular bisector of AD will intersect this line at a unique point, it suffices to show $Y'A = Y'D$. From $AXY'D$ cyclic and the angle bisector we have

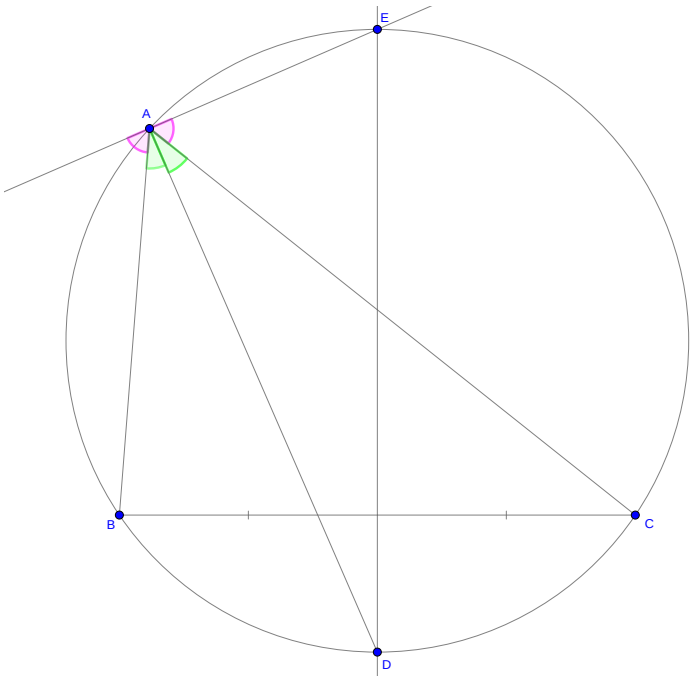
$$\angle Y'DA = \angle Y'XC = \angle DXY' = \angle DAY'.$$

Thus triangle $Y'AD$ has base angles equal and is isosceles with $Y'A = Y'D$ as required. A similar argument shows $BCXY$ is cyclic.

Remark

A useful fact (often phrased in terms of arc midpoints) is the following.

Let ABC be a triangle with circumcircle Γ, and let the perpendicular bisector of BC meet Γ at different points D and E where E lies on the same side of BC as A. Then AD is the internal bisector of $\angle BAC$, while AE is the external bisector.



We prove the external form of this result in the solution above and the more commonly used internal form has a similar proof.

This can be used to prove a quadrilateral is cyclic by showing a point D lies both on the perpendicular bisector of BC and an angle bisector (either internal or external) of $\angle BAC$.

Remark

The solution above can also be phrased in terms of spiral similarities. This is explained in detail in the video solution to this problem which is available on the UKMT website.

3. Each cell of a 30×30 grid contains one of the numbers -1, 0 or 1 with each of these three numbers appearing exactly 300 times.

Is it possible that the 60 row and column sums are all different?

SOLUTION

We argue by contradiction. Suppose that it is possible for the 30 row sums and 30 column sums to be all different.

Each row sum and column sum is an integer between -30 and 30, so there are 61 possible values. Since we have 60 distinct line sums, exactly one integer in $\{-30, -29, \ldots, 29, 30\}$ does not occur. The total of both the entries in the grid and the elements of this set are 0 so the sum that is omitted must be 0. Therefore the 60 line-sums are precisely

$$\pm 1, \pm 2, \ldots, \pm 30.$$

In particular, there are 30 positive line-sums and 30 negative line-sums.

Let n be the number of rows with positive sum. Then there are $30 - n$ rows with negative sum. Also, since there are 30 positive line-sums in total, the number of columns with positive sum must be $30 - n$, and so the number of columns with negative sum is n.

Now consider

$$T = \sum_{\text{rows}} |(\text{row sum})| + \sum_{\text{columns}} |(\text{column sum})|.$$

Since the line-sums are $\pm 1, \pm 2, \ldots, \pm 30$, we have

$$T = 2(1 + 2 + \cdots + 30) = 2 \cdot \frac{30 \cdot 31}{2} = 930.$$

We obtain an upper bound for T by looking at the contribution from each cell. For a row with positive sum we keep its sum as it is; for a row with negative sum we take the negative of its sum. Doing this for all rows produces exactly the total contribution $\sum_{\text{rows}} |(\text{row sum})|$. Doing the same for the columns, and adding the results gives T.

When we expand this total as a sum over all 900 cells, a cell in a row and a column of opposite signs is counted once with a + sign and once with a $-$ sign, so its contributions cancel and it contributes 0 to T.

A cell in a positive row and a positive column, or in a negative row and a negative column, is counted twice with the same sign. Thus such a cell contributes either 2, 0 or -2 to T, and in particular it contributes at most 2.

There are $n(30 - n)$ cells in positive rows and positive columns, and also $n(30 - n)$ cells in negative rows and negative columns, so

$$T \leq 2(n(30 - n) + n(30 - n)) = 4n(30 - n).$$

Finally, by the AM-GM inequality,

$$n(30 - n) \leq \left(\frac{n + (30 - n)}{2}\right)^2 = 15^2 = 225.$$

© 2026 UK Mathematics Trust

This gives $T \leq 4 \cdot 225 = 900$, contradicting $T = 930$.

This contradiction shows that it is not possible for all 60 row and column sums to be different.

ALTERNATIVE

Consider a $3n \times 3n$ grid containing $3n^2$ copies of each of $-1, 0, 1$ where all $6n$ row and column sums are different. We again argue by contradiction. As before, the line-sums are $\pm 1, \pm 2, \ldots, \pm 3n$.

Focus on the $2n$ lines whose sums are $n+1, n+2, \ldots, 3n$. A line with sum $s > 0$ must contain at least s entries equal to 1 (since every -1 reduces the sum). Therefore, if we count 1s with multiplicity across these $2n$ lines, we count at least

$$(n+1) + (n+2) + \cdots + 3n = n(4n+1)$$

ones.

Among these $2n$ lines, let r be the number of rows selected; then $2n - r$ columns are selected. A 1 can be counted twice only if it lies in the intersection of a selected row and a selected column, and there are

$$r(2n-r) \leq \left(\frac{r + (2n-r)}{2}\right)^2 = n^2$$

such cells, where we apply AM-GM to get the bound. Hence the number of *distinct* 1s in the union of these $2n$ lines is at least

$$n(4n+1) - n^2 = 3n^2 + n,$$

which is greater than $3n^2$. This is impossible, because the entire grid contains exactly $3n^2$ entries equal to 1.

Taking $n = 10$ gives the original 30×30 problem, and yields the required contradiction.

© 2026 UK Mathematics Trust

4. Let N be a positive integer and let $(k_n)_{n\geq 1}$ be a sequence of positive integers with all terms at most N. Annabel begins by choosing integers $x_1, x_2, \ldots, x_N$. She then extends this to an infinite sequence $(x_n)_{n\geq 1}$ of integers by defining

$$x_n = \sum_{i=n-k_n}^{n-1} x_i$$

for each $n > N$.

Show that there are either finitely many strictly positive terms or finitely many strictly negative terms in the infinite sequence (x_n).

SOLUTION

We prove the result by contradiction. Assume there are infinitely many strictly positive terms and infinitely many strictly negative terms. Define sequences (a_n) and (b_n) for $n > N$ as follows:

$$\begin{aligned} a_n &= \min\left\{\sum_{i=n-k}^{n-1} x_i : 1 \leq k \leq N\right\}; \\ b_n &= \max\left\{\sum_{i=n-k}^{n-1} x_i : 1 \leq k \leq N\right\}. \end{aligned}$$

Since x_n is one of the terms in the set, we have $a_n \leq x_n \leq b_n$. Also, each term that appears in the set for a_{n+1} except x_n, is a term that appeared in the set for a_n with x_n added. Hence

$$a_{n+1} \geq \min\{x_n, a_n + x_n\}.$$

If $a_n \geq 0$ for some n, then $x_n \geq a_n \geq 0$ and, by the above, $a_{n+1} \geq 0$. Thus, from this point on, all terms x_i would be non-negative, contradicting our assumption that there are infinitely many negative terms x_i. Therefore, we may assume that $a_n < 0$ for all $n > N$.

Similarly, $b_{n+1} \leq \max\{x_n, b_n + x_n\}$ and, by a similar argument, we may assume $b_n > 0$ for all $n > N$. Define $d_n = b_n - a_n$ for $n > N$. We now prove two claims.

Claim 1: For $n > N$, (d_n) is non-increasing.

Proof. Since $a_n \leq 0 \leq b_n$, we have that

$$\begin{aligned} a_{n+1} &\geq \min\{x_n, a_n + x_n\} = a_n + x_n; \\ b_{n+1} &\leq \max\{x_n, b_n + x_n\} = b_n + x_n. \end{aligned}$$

Combining these

$$d_{n+1} = b_{n+1} - a_{n+1} \leq (b_n + x_n) - (a_n + x_n) = b_n - a_n = d_n.$$

□

Since d_n is bounded below (by 0) and is a sequence of integers, it follows from Claim 1 that (d_n) is eventually constant so for some positive integer M, we have $d_n = d$ for all $n > M$.

Since $d_n = d$ is constant for $n > M$, we must have equality throughout the proof of Claim 1 and in particular, $b_{n+1} = b_n + x_n$ for $n > M$. Define $K_n \geq 1$ for $n > M$ to be the smallest positive integer such that

$$b_n = \sum_{i=n-K_n}^{n-1} x_i.$$

By the definition of b_n, we have $K_n \leq N$.

Claim 2: $K_{n+1} = K_n + 1$ for $n > M$.

Proof. Firstly, observe we have

$$b_{n+1} = x_n + b_n = x_n + \sum_{i=n-K_n}^{n-1} x_i = \sum_{i=n-K_n}^{n} x_i = \sum_{i=(n+1)-(K_n+1)}^{n} x_i$$

so $K_n + 1$ satisfies the required condition. The only way for $K_{n+1} \neq K_n + 1$ is if there exists some $L < K_n + 1$ with

$$b_{n+1} = x_n + b_n = \sum_{i=n+1-L}^{n} x_i \implies b_n = \sum_{i=n-(L-1)}^{n-1} x_i.$$

However, as $L - 1 < K_n$, this contradicts the minimality of K_n unless $L = 1$, but that case does not arise because then $b_n = 0$. □

Thus the sequence (K_i) is strictly increasing, but all terms are at most N. This is absurd and the proof is complete.

Remark

The condition that (x_n) consists of integers is essential, as shown by the sequence $x_n = \left(\frac{1-\sqrt{5}}{2}\right)^n$ which satisfies $x_{n+2} = x_{n+1} + x_n$ and contains infinitely many positive and negative terms.