UK Maths Trust

Senior Mathematical Challenge

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For reasons of space, these solutions are necessarily brief.
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1. E Let the two numbers be x and y with $x > y$. Therefore $x + y = 20$ and $x - y = 26$. Adding these equations gives $2x = 46$, so $x = 23$ and $y = -3$. The product is $23 \times -3 = -69$.

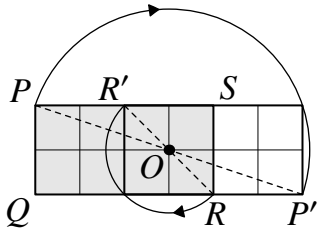
2. C Let a pence be the price of an apple and p pence be the price of a pear. Therefore $4a + 5p = 320$ and $5a + 4p = 310$. Adding these equations gives $9a + 9p = 630$, therefore the total cost of one apple and one pear is 70 pence.

3. B Let the side-length of the square be x and the side-length of the rhombus be y. As the areas are equal, we have $x^2 = y \times y \times \sin 30°$. Therefore $x^2 = y^2 \times \frac{1}{2}$ which rearranges to $\sqrt{2}x = y$. The required ratio is then $1 : \sqrt{2}$.

4. D Since $2^x = \sqrt{4\sqrt{4\sqrt{4}}}$, squaring both sides gives $2^{2x} = 4\sqrt{4 \times 2}$. Expressing both sides as powers of 2, $2^{2x} = 2^2 \times 2^{\frac{3}{2}}$. Equating the powers, $2x = 2 + \frac{3}{2}$ and therefore $x = \frac{7}{4}$.

5. A In order to create a sequence for which the mean of the first n terms is $\frac{1}{n}$, the sum of the first n terms must be 1. The sequence must therefore be 1, 0, 0, 0, 0, ... , with first term 1 and all subsequent terms, including the 1000th term, having value 0.

6. D Both P and R travel the length of a semicircle during the rotation of 180°. Using Pythagoras' Theorem, the length of OR is $\sqrt{2}$, so R moves $\pi\sqrt{2}$ to reach R'. Similarly, the length of OP is $\sqrt{10}$, so P moves $\pi\sqrt{10}$ to reach P'. Therefore $k = \frac{\pi\sqrt{10}}{\pi\sqrt{2}} = \sqrt{5}$.



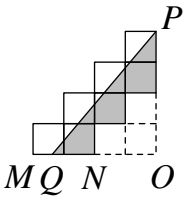
7. D In order to have more purple beads than yellow in a group of 15 beads, we require there to be 8 or more purples. As there are 25 purples available and $25 = 3 \times 8 + 1$, it is possible to create just 3 groups with more purples than yellows.

8. B The largest number that can be created with ten different digits is 9876543210, which by division can be shown to be 2 more than a multiple of 7. Swapping any two adjacent digits in the large number reduces its value by a multiple of 9. Swapping the tens and ones digits reduces the number by exactly 9, which in combination with the original remainder of 2 now produces a number $N = 9876543201$ which is a multiple of 7. The remainder when N is divided by ten is then 1.

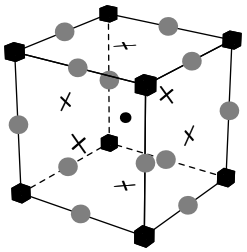
9. **B** The number of 'a's on line k is the sum of the number of 'a's on line $k-1$ and the number of 'b's on line $k-1$. The latter is equivalent to the number of 'a's on line $k-2$. As line 1 is a single 'a', the numbers of 'a's at each stage are the Fibonacci numbers. We therefore require the 10th Fibonacci number which is 55.

10. **C** Building a seven-digit number from the left, first notice that both a 4 and a 6 must be followed by a 5 (one choice). However a 5 can be followed by either a 4 or a 6 (two choices). Therefore there are $1 \times 2 \times 1 \times 2 \times 1 \times 2 \times 1 = 8$ possible seven-digit numbers which start with a 5. Similarly there are $2 \times 1 \times 2 \times 1 \times 2 \times 1 \times 2 = 16$ possible seven-digit numbers which start with a 4 or a 6. In total, there are $8 + 16 = 24$ numbers which fit the constraints given.

11. **C** In order to find the position of Q, we can consider the area of triangle QOP which is made from half of the original 7 squares plus the 3 extra, shown with dotted lines in the diagram. Therefore $\frac{1}{2}OQ \times OP = \frac{13}{2}$. Since $OP = 4$, $OQ = \frac{13}{4}$. Then $MQ = 4 - \frac{13}{4} = \frac{3}{4}$ and $NQ = \frac{5}{4}$. The ratio $MQ : QN = 3 : 5$



12. **B** Dots can either be at a vertex (type V), on the midpoint of an edge (type E) or at the midpoint of a face (type F). The diagram shows these using different symbols in order to distinguish between them. Since there is no dot in the centre of the cube, any line of three dots must lie entirely on one face of the cube. The lines must therefore be of the form VEV, VFV or EFE. Type VEV lie along each edge of which there are 12, type VFV lie on each diagonal of the six faces (two per face) and type EFE lie on each of the six faces (again, two per face). In total there are $12 + 2 \times 6 + 2 \times 6 = 36$ possible lines.

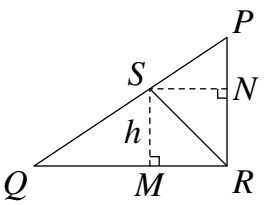


13. **E** Since $\angle QMP = 60°$, being inside equilateral triangle KMN, and $\angle QPM = 90°$ (given), the side-lengths of triangle QPM are in the ratio $1 : \sqrt{3} : 2$. As the length PQ is given as $t\sqrt{3}$, length $PM = t$ and length $QM = 2t$. We are also given that PN and QJ are the same length; let this be x. As MN has length 26, $t + x = 26$. As JK has length 20, JM has length 46 and so $2t + x = 46$. The difference between these equations gives $t = 20$.

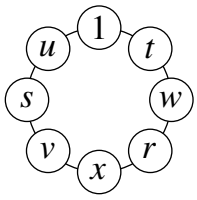
14. **C** Expressing each fraction using a common denominator of 60 gives $\frac{60}{60}, \frac{30}{60}, \frac{20}{60}, \frac{15}{60}, \frac{12}{60}, \frac{10}{60}$. Adding them all produces the largest total of $\frac{147}{60}$. In order to produce the smallest possible positive value we need to partition the fractions into two sets whose totals are as close as possible to half of this. The value sought is the difference between these totals. By inspection of the six numerators, this is achieved by $\frac{75-72}{60} = \frac{3}{60} = \frac{1}{20}$.

15. **A** Since the mean of n numbers is n, the total of those n numbers is n^2. Two of the numbers have a total of $4n + (n-6) = 5n - 6$, therefore the remaining $(n-2)$ numbers, each of whose value is k, have a total of $n^2 - 5n + 6$. Hence $k(n-2) = n^2 - 5n + 6$ and factorising the RHS as $(n-3)(n-2)$ gives $k = n - 3$.

16. **B** Drawing perpendiculars from S to QR and PR produces a quadrilateral $SMRN$. Since $\angle QRS = 45°$, $SMRN$ is a square. Let the length of each side of the square be h. The area of triangle PQR is equal to the sum of the areas of triangles SQR and PSR. Hence $\frac{1}{2} \times 3\sqrt{5} \times 2\sqrt{5} = \frac{1}{2}3\sqrt{5} \times h + \frac{1}{2}2\sqrt{5} \times h$ which simplifies to $15 = \frac{5\sqrt{5}}{2}h$ and $h = \frac{6}{\sqrt{5}}$. Therefore the area of triangle $PSR = \frac{1}{2}2\sqrt{5} \times \frac{6}{\sqrt{5}} = 6$.



17. **C** Let the values in the eight circles be 1, r, s, t, u, v, w, and x as shown. In order that the numbers in adjacent pairs of circles have a sum which is prime, r, u, v and t must be even and s, w and x must be odd. Consider first the position of the number 8. Since 8 may not be next to 1 (9 is not prime), either $r = 8$ or $v = 8$. Without loss of generality, choose $r = 8$. Consider next the position of the number 7. Since 7 may not be next to 8 (15 is not prime), $s = 7$. Consider next the position of the number 2. Since 2 may not be next to 7 (9 is not prime), $t = 2$.



So $u = 4$ or 6. If $u = 4$ then $v = 6$, therefore $x = 5$ (to avoid $6 + 3 = 9$) and $w = 3$. If however $u = 6$ then $v = 4$, therefore $x = 3$ (to avoid $4 + 5 = 9$) and $w = 5$. There are therefore two possible values of x which allow the diagram to be completed.

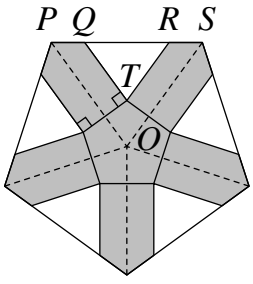
18. **A** First note that the sum of the first n odd numbers $1+3+\ldots+(2n-1)$ is the square number n^2. The sum of the first n even numbers is n more than that, which is $n^2 + n = n(n+1)$. Now considering function g, $g(a) \times g(b^2) \times g(a^3) \times g(b^4) \times \ldots \times g(a^{2n-1}) \times g(b^{2n}) = a^{\frac{1}{n}} b^{\frac{2}{n}} a^{\frac{3}{n}} b^{\frac{4}{n}} \times \ldots \times a^{\frac{2n-1}{n}} b^{\frac{2n}{n}}$ $= a^{(\frac{1}{n}+\frac{3}{n}+\ldots+\frac{2n-1}{n})} \times b^{(\frac{2}{n}+\frac{4}{n}+\ldots+\frac{2n}{n})}$. Using the earlier results this is equal to $a^{\frac{n^2}{n}} \times b^{\frac{n(n+1)}{n}} = a^n b^{n+1}$.

19. **A** Considering the sequence of positive integers x, the first value of x for which $\lfloor x^{\frac{1}{2}} \rfloor = k$ for some integer k, is a square. The last value of x for which $\lfloor x^{\frac{1}{2}} \rfloor = k$ is one less than the next square. For example, $\lfloor x^{\frac{1}{2}} \rfloor = 6$ when $36 \le x < 49$. Similarly, $\lfloor x^{\frac{1}{3}} \rfloor = 2$ when $8 \le x < 27$. This is a case where $6 + 2 = 8$ but there is no overlap between the two sets of values of x. The only sets which have values of x in common and for which $\lfloor x^{\frac{1}{2}} \rfloor + \lfloor x^{\frac{1}{3}} \rfloor = 8$ are $\lfloor x^{\frac{1}{2}} \rfloor = 5$, when $25 \le x < 36$ and $\lfloor x^{\frac{1}{3}} \rfloor = 3$, when $27 \le x < 81$. The nine common values of x are 27, 28, 29, 30, 31, 32, 33, 34 and 35.

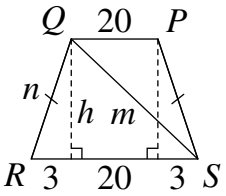
20. **E** By factorising $y = 1 + \sin x \cos x - \sin x - \cos x$ as $y = (1 - \cos x)(1 - \sin x)$ we observe that $y = 0$ when $1 - \cos x = 0$, so $\cos x = 1$ which is so when $x = 0$ and when $x = 360$. Also, we have another point where $y = 0$ when $1 - \sin x = 0$, so $\sin x = 1$ which is so when $x = 90$. The graph in option E is the only one with all of these properties.

21. **E** The eight vertices of the cube can be split into two sets, (1) containing Q, S, U and W and (2) containing P, R, T and V. When the ant travels along the diagonal of a face, it moves to a vertex in the same set. When the ant travels along an edge, it moves to a vertex in the opposite set. As a result of the alternating nature of the ant's journey, the sets which contain the vertices at which the ant could be produce a sequence of period 4; (1), (2), (2), (1), (1), (2), (2), (1), ... Since 81 is 1 more than a multiple of 4, the ant can only be at a vertex within (1), and so it can not be at vertex V.

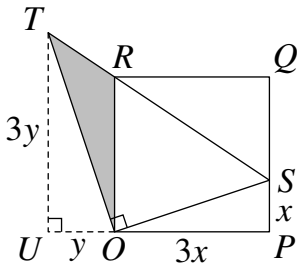
22. **D** The diagram can be split into five congruent triangles along the dotted lines as shown. At point T, the angle inside the smaller regular pentagon is 108°. Also, $\angle QTR = 360° - (2 \times 90° + 108°) = 72°$. Therefore $\angle QTR = \angle POS$ and hence triangle QTR and triangle POS are similar as their bases lie along the same edge. Since $PS = 10$ and $QR = 10 - 2 \times 2 = 6$, the lengths QR and PS are in the ratio 3 : 5. The areas of triangles QTR and POS are therefore in the ratio $3^2 : 5^2$. The fraction of the area of the playground which is shaded is then $1 - \frac{9}{25} = \frac{16}{25}$.



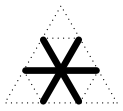
23. **129** Consider the two right-angled triangles shown, with height h and vertex Q. Applying Pythagoras' Theorem, first to the triangle with hypotenuse QR gives $3^2 + h^2 = n^2$ (1) and then to the triangle with hypotenuse QS gives $23^2 + h^2 = m^2$ (2). Subtracting, (2) − (1) gives $23^2 - 3^2 = m^2 - n^2$. Therefore $520 = (m+n)(m-n)$. As both $m+n$ and $m-n$ are integers, we can consider the factor pairs of 520. Taking first $m+n = 520$ and $m-n = 1$ produces non-integer values of m and n. Moving to $m+n = 260$ (3) and $m-n = 2$ (4) however leads to $2n = 258$ and therefore $n = 129$. Any other factor pairs will have values closer together than 260 and 2 and so will produce a smaller value of n.



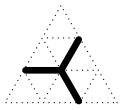
24. **140** Since $PS : SQ = 1 : 2$, let $PS = x$ and therefore $OP = 3x$. The length of $OS = \sqrt{10}x$ and the gradient of OS is $\frac{x}{3x} = \frac{1}{3}$. Since $\angle TOS = 90°$, the gradient of OT is -3. Extending PO until point U, directly below T, allows us to show that if $OU = y$ then $UT = 3y$. Therefore the length of $OT = \sqrt{10}y$. We can now express the shaded area in terms of x and y in two ways. Firstly, viewed as a triangle with base OR and perpendicular height OU, its area is $\frac{1}{2} \times 3x \times y$. Secondly, viewed as the triangle TOS with the area of ROS (base OR and perpendicular height OP) removed, its area is $\frac{1}{2} \times \sqrt{10}x \times \sqrt{10}y$ - $\frac{1}{2} \times 3x \times 3x$. Equating these two expressions gives $\frac{3xy}{2} = 5xy - \frac{9x^2}{2}$ and therefore $9x^2 = 7xy$. We are given that the shaded area is 30, so $\frac{1}{2} \times 3x \times y = 30$ and $xy = 20$. The area of $OPQR$ which is $9x^2$ is therefore $7 \times 20 = 140$.



25. **528** The problem reduces to the number of ways of colouring the central six lines. Once their colours are chosen, the colours of the extra six lines which close the six central triangles, shown dotted, are uniquely determined. Then there are two ways to colour each of the corner triangles, so $2^3 = 8$ ways once the central six triangles are coloured.



When considering the colouring of the central six lines, one successful strategy is to split the counting into three cases:



(a) The three spokes shown are differently coloured. There are 3! = 6 ways to do this. In each case, the rest of the colouring of the central hexagon happens uniquely.

(b) The three spokes are all the same colour. There are 3 ways this can happen.The other 3 spokes can then be coloured in 2^3 ways and then the rest of the colouring of the central hexagon happens uniquely.

(c) Two spokes are one colour, and one spoke is another colour. There are 3 ways to choose the positions of the identically coloured spokes, and then 3×2 ways to choose the colours used. Now consider the other three spokes. The colours of two of them are forced, and the third could be one of two possible colours. The rest of the colouring of the central hexagon is then uniquely determined.

This gives $8 \times (6 + (3 \times 2^3) + (3 \times 3 \times 2 \times 2)) = 8 \times 66 = 528$ different ways.