UK Maths Trust

# Senior Mathematical Challenge

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## Solutions and investigations

**7 October 2026**

These solutions augment the shorter solutions also available online. The shorter solutions sometimes leave out details. The solutions given here are full solutions, as explained below. In some cases alternative solutions are given. There are also many additional problems for further investigation. We welcome comments on these solutions and the additional problems. Please send them to challenges@ukmt.org.uk.

The Senior Mathematical Challenge (SMC) is a multiple-choice paper. Sometimes you can find the correct answers by working backwards from the given options, or by showing that four of them are not correct. This can be a sensible thing to do in the context of the SMC.

However, this does not provide a full mathematical explanation that would be acceptable if you were just given the questions without any alternative answers. So we aim at including for each question a complete solution with each step explained (or, occasionally, left as an exercise), and not based on the assumption that one of the given options is correct. We hope that these solutions provide a model for the type of written solution that is expected when presenting a complete solution to a mathematical problem (for example, in the British Mathematical Olympiad, the Mathematical Olympiad for Girls and similar competitions).

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| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| E | C | B | D | A | D | D | B | B | C | C | B | E | C | A | B | C | A | A | E | E | D | 1 | 2 | 9 | 1 | 4 | 0 | 5 | 2 | 8 |

**1.** What is the product of two numbers whose sum is 20 and whose difference is 26?

A 6 B 46 C 12 D −46 E −69

SOLUTION **E**

Let the two numbers be $x$ and $y$. Then, by what we are told in the question,

$$x + y = 20, \qquad (1)$$

and

$$x - y = 26. \qquad (2)$$

Adding equations (1) and (2) gives $2x = 46$, from which it follows that $x = 23$. Hence, by equation (1), $y = 20 - x = 20 - 23 = -3$.

We conclude that the product $x \times y$ is equal to $23 \times -3 = -69$.

FOR INVESTIGATION

**1.1** (a) Prove that $(x + y)^2 - (x - y)^2 = 4xy$, for all numbers $x$ and $y$.

(b) Use the result of part (a) to answer Question 1.

(c) Find the product of two numbers whose sum is $\sqrt{3}$ and whose difference is $\sqrt{2}$.

(d) Two numbers have sum 5 and product 3.

What is their difference?

**2.** Alana buys 4 apples and 5 pears for £3.20. At the same prices, 5 apples and 4 pears cost £3.10.

What is the total cost of one apple and one pear?

A 50p B 60p C 70p D 80p E 90p

SOLUTION **C**

(4 apples and 5 pears) + (5 apples and 4 pears) cost £3.20 + £3.10 = £6.30. That is, 9 apples and 9 pears cost £6.30. Hence 1 apple and 1 pear cost £6.30 ÷ 9 = 70p,

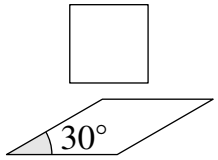
FOR INVESTIGATION

**2.1** Three apples, five pears and seven oranges cost £8.10. One apple, two pears and three oranges cost £3.25.

What is the cost of one apple, one pear and one orange?

**3.** The square and the rhombus with interior angle of 30° have the same area.

What is the ratio of the side-length of the square to the side-length of the rhombus?




A 1 : 1 B 1 : $\sqrt{2}$ C $\sqrt{3}$ : 2 D 2 : 3 E 1 : 2

SOLUTION **B**

Let the side length of the square be $s$ and the side length of the rhombus be $r$.

The area of the square is $s^2$. The area of the rhombus is $r^2 \sin(30°) = \frac{1}{2}r^2$. [You are asked to prove this in Problem 3.1.]

Since these are equal, $s^2 = \frac{1}{2}r^2$. Hence $s = \frac{1}{\sqrt{2}}r$. Therefore $s : r = 1 : \sqrt{2}$.

FOR INVESTIGATION

**3.1** A rhombus has sides of length $r$. The acute angle between its sides is $\theta$. Explain why the area of the rhombus is $r^2 \sin\theta$.

**3.2** (a) Find the ratio of the length of the long diagonal of the rhombus of this question to the length of its sides.

(b) Find an exact expression for the value of $\sin(15°)$.

**4.** $2^x = \sqrt{4\sqrt{4\sqrt{4}}}$. What is the value of $x$?

A 3 B $\frac{5}{2}$ C 2 D $\frac{7}{4}$ E $\frac{3}{2}$

SOLUTION **D**

We have

$$\begin{aligned}\sqrt{4\sqrt{4\sqrt{4}}} &= \sqrt{2^2\sqrt{2^2\sqrt{2^2}}} = \sqrt{2^2\sqrt{2^2 \times 2}} \\ &= \sqrt{2^2\sqrt{2^3}} = \sqrt{2^2 \times 2^{\frac{3}{2}}} \\ &= \sqrt{2^{\frac{7}{2}}} = 2^{\frac{7}{4}}.\end{aligned}$$

Hence $x = \frac{7}{4}$.

FOR INVESTIGATION

**4.1** $3^x = \sqrt{9\sqrt{9\sqrt{9\sqrt{9}}}}$. What is the value of $x$?

**5.** A sequence of numbers has the property that for every positive integer $n$, the mean of the first $n$ terms of the sequence is $\dfrac{1}{n}$.

What is the 1000th term of this sequence?

A 0 B $\frac{1}{999}$ C $\frac{1}{1000}$ D $\frac{1}{1001}$ E $\frac{1}{1000000}$

SOLUTION **A**

For every positive integer $n$, since the mean of the first $n$ terms of the sequence is $\frac{1}{n}$, the sum of the first $n$ terms is $n \times \frac{1}{n} = 1$.

Since this is true for $n = 1$, the first term of the sequence is 1. Since it is also true for $n > 1$, all the remaining terms are 0. In particular, the 1000th term of the sequence is 0.

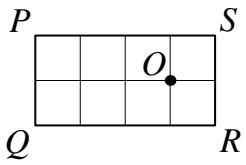
FOR INVESTIGATION

**5.1** Another sequence of numbers has the property that for every positive integer $n$ the mean of the first $n$ terms of the sequence is $n$.

(a) What is the 1000th term of this sequence?

(b) Find an expression in terms of $n$ for the $n$th term of this sequence.

**6.** Rectangle $PQRS$ is made from eight 1 by 1 squares as shown in the diagram. At point $O$, four of those squares meet. The rectangle is rotated through 180° about point $O$. Point $P$ travels $k$ times as far as point $R$ during that rotation.




What is the value of $k$?

A $\sqrt{2}$ B $\sqrt{3}$ C 2 D $\sqrt{5}$ E $\sqrt{6}$

SOLUTION **D**

The point $P$ travels round a semicircle whose radius is the length of $OP$. By Pythagoras' theorem this radius is $\sqrt{1^2 + 3^2} = \sqrt{10}$,

The point $R$ travels round a semicircle whose radius is the length of $OR$. By Pythagoras' theorem this radius is $\sqrt{1^2 + 1^2} = \sqrt{2}$.

The length of a semicircle is $\pi r$, where $r$ is its radius. Hence we have,

$$\frac{\text{distance travelled by } P}{\text{distance travelled by } R} = \frac{\pi\sqrt{10}}{\pi\sqrt{2}} = \sqrt{5}.$$

FOR INVESTIGATION

**6.1** Suppose that the rectangle $PQRS$ is is rotated through 360° about the point $Q$. Each of the points $O$, $P$, $R$ and $S$ moves round in a complete circle.

Find the areas of these circles.

**7.** A bag contains 50 yellow and 25 purple beads which are to be arranged into five groups of 15.

What is the largest possible number of the groups having more purple than yellow beads?

A 0 B 1 C 2 D 3 E 4

SOLUTION **D**

The smallest number of purple beads needed to ensure that a group of 15 yellow and purple beads contains more purple beads than yellow beads is 8, with the other 7 being yellow..

Since there are 25 purple beads and $25 \div 8 = 3\frac{1}{8}$, the largest possible number of groups containing 8 purple beads is 3.

So the largest possible number of groups with more purple beads than yellow beads is 3.

FOR INVESTIGATION

**7.1** What is the largest possible number of the groups having more than twice as many yellow beads as purple beads?

**8.** The number $N$ is the largest 10-digit integer, all of whose digits are different, which is divisible by 7.

What is the remainder when $N$ is divided by 10?

A 0 B 1 C 2 D 3 E 4

SOLUTION **B**

The largest 10-digit number all of whose digits are different is 9 876 543 210.

The following long division sum

$$\begin{array}{r} 1\ 4\ 1\ 0\ 9\ 3\ 4\ 7\ 4\ 4 \\ 7\overline{)9^{2}8\ 7\ 6\ ^{6}5^{2}4^{3}3^{5}2^{3}1^{3}0^{2}} \end{array}$$

shows that $9\,876\,543\,210 \div 7 = 1\,410\,934\,744$ with remainder 2.

It follows that $9\,876\,543\,210 - 2 = 9\,876\,543\,208$ is divisible by 7. Unfortunately, its digits are not all different. However, $9\,876\,543\,208 - 7 = 9\,876\,543\,201$, is also divisible by 7, and all its digits are different.

Since 9 876 543 201 is the second largest 10-digit number all of whose digits are different and it is divisible by 7, it is the largest 10-digit integer all of whose digits are different which is divisible by 7.

The remainder when a positive integer is divided by 10 is its units digit. Therefore the remainder when $N = 9\,876\,543\,201$ is divided by 10 is 1.

FOR INVESTIGATION

**8.1** Which is the smallest 10-digit integer, all of whose digits are different, which is divisible by 7?

**9.** A string of letters is created by applying the following rule in each iteration: Any '$a$' is replaced by '$ab$', and any '$b$' is replaced by letter '$a$'. Alison writes down letter '$a$' in line 1, then she applies the rule and writes down '$ab$' in line 2, and '$aba$' in line 3.

How many '$a$'s will appear in line 10 if Alison keeps applying the same rule correctly?

A 34 B 55 C 64 D 89 E 128

SOLUTION **B**

At each stage Alison replaces '$a$' by '$ab$' and '$b$' by '$a$'. Therefore, the number of '$a$'s on the each line, after the first, will equal the the total number of '$a$'s and '$b$'s on the previous line. Similarly, the number of '$b$'s will be the same as the number of '$a$'s on the previous line.

We can therefore draw up the following table.

| line | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| number of '$a$'s | 1 | 1 | 2 | 3 | 5 | 8 | 13 | 21 | 34 | 55 |
| number of '$b$'s | 0 | 1 | 1 | 2 | 3 | 5 | 8 | 13 | 21 | 34 |

From this we see that in line 10 there are 55 '$a$'s.

FOR INVESTIGATION

**9.1** The numbers giving the number of '$a$'s in each line in the table above are the first ten terms of a well known sequence.

(a) Which well known sequence is it?

(b) Prove that the sequence of numbers showing the number of '$a$'s in each line, when continued indefinitely, coincides with this well known sequence.

**9.2** A sequence of strings of letters is created using the following rules:

(1) The sequence '$a$' is on line 1.

(2) In moving from one line to the next '$a$' is replaced by '$aab$' and '$b$' is replaced by '$ab$'.

(a) Draw up a table giving the number of '$a$'s and '$b$'s in the first ten lines.

(b) What do you notice about the sequence of numbers that occur in your table?

(c) Prove that if the table is continued indefinitely, this pattern continues.

**10.** How many seven-digit numbers are there such that each digit is either 4, 5 or 6 and each pair of adjacent digits differ by 1?

A 12 B 18 C 24 D 30 E 32

SOLUTION **C**

Because each pair of adjacent digits differ by 1, the digits 4 and 6 must be followed by 5, whereas 5 can be followed by either 4 or 6.

It follows that the seven-digit numbers that meet the required condition have either the form *5*5*5* or 5*5*5*5 where, in each case, * can be either 4 or 6.

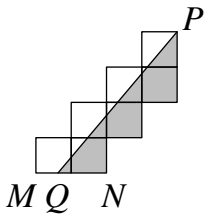
Since there are two choices for * there are $2 \times 2 \times 2 \times 2 = 16$ numbers of the form *5*5*5* and $2 \times 2 \times 2 = 8$ of the form 5*5*5*5.

This makes a total of $16 + 8 = 24$ numbers of the required form.

FOR INVESTIGATION

**10.1** How many nine-digit numbers are there such that each digit is 1, 2, 3 or 4 and each pair of adjacent digits differ by 1 or by 3?

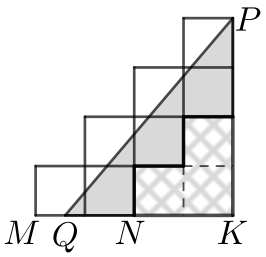
**11.** Seven squares of unit length are aligned edge-to-edge as shown. A straight line is drawn from $P$ to $Q$, where $Q$ is the point on the line $MN$ such that the shaded area formed is half the total area of the seven squares. What is the ratio $MQ : QN$?




A 1 : 2 B 2 : 3 C 3 : 5 D 5 : 8 E 8 : 13

SOLUTION **C**

Let the length of $QN$ be $x$, and let $K$ be the point where the line through $P$ perpendicular to $MN$ meets the extension of $MN$, as shown in the diagram.




Then $QK$ has length $x + 2$ and $PK$ has length 4. Hence the triangle $PKQ$ has area $\frac{1}{2}(QK \times PK) = \frac{1}{2}((x + 2) \times 4) = 2(x + 2)$.

The shaded area is the area of the triangle $PKQ$ less the area of the three squares shown cross-hatched in the diagram.

It follows that the shaded area is $2(x + 2) - 3 = 2x + 1$. This is half the total area of the seven squares. Therefore $2x + 1 = \frac{7}{2}$. Hence $2x = \frac{7}{2} - 1 = \frac{5}{2}$. Therefore $x = \frac{5}{4}$.

Therefore $MQ = MN - QN = 2 - \frac{5}{4} = \frac{3}{4}$. Hence $MQ : QN = \frac{3}{4} : \frac{5}{4} = 3 : 5$.

FOR INVESTIGATION

**11.1** What fraction of the seven squares would be shaded if $MQ : QN = 4 : 7$?

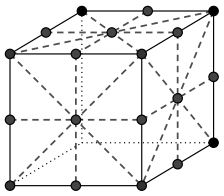
**12.** A cube has a dot at each vertex, at the midpoint of each edge and at the midpoint of each face.

How many straight lines can pass through exactly 3 dots?

A 24 B 36 C 48 D 60 E 72

SOLUTION **B**

Each of the 12 edges of the cube is a straight line which goes through exactly 3 dots.



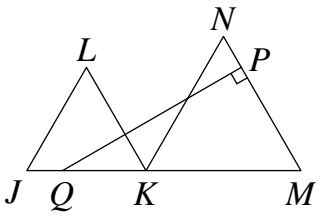
In addition, on each of the 6 faces of the cube there are 4 straight lines which go through 3 dots. These are shown as dashed lines in the diagram on the right.

There are no other straight lines that go through exactly 3 dots. Therefore the total number of these lines is $12 + 6 \times 4 = 36$.

FOR INVESTIGATION

**12.1** How many equilateral triangles and how many isosceles triangles are there each of which hass three of the dots as its vertices?

**13.** $JKL$ and $KMN$ are equilateral triangles with side-lengths 20 and 26 respectively. Points $J$, $K$ and $M$ lie on a straight line. Points $P$ on $NM$, and $Q$ on $JK$, are such that $JQ$ and $NP$ have the same length and $QPM$ is a right angle. The length of $PQ$ is $t\sqrt{3}$.




What is the value of $t$?

A 24 B 23 C 22 D 21 E 20

SOLUTION **E**

Let $JQ = NP = x$. We then have $QM = JK + KM - JQ = 20 + 26 - x = 46 - x$, and $PM = NM - NP = 26 - x$.

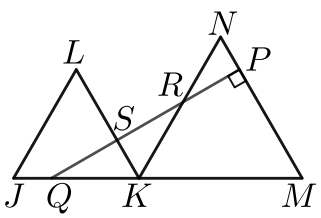
$QPM$ is a right-angled triangle in which $\angle QMP = 60°$. Therefore $PM = QM \cos 60° = \frac{1}{2}QM$. That is, $26 - x = \frac{1}{2}(46 - x)$. It follows that $52 - 2x = 46 - x$. Therefore $x = 6$.

Hence $QM = 40$. It follows that $PQ = 40 \sin 60° = 40\frac{\sqrt{3}}{2} = 20\sqrt{3}$. Therefore $t = 20$.

FOR INVESTIGATION

**13.1** Let $R$ be the point where the line $PQ$ meets $NK$, and let $S$ be the point where $PQ$ meets $LK$.

Show that $RS = SQ$ and find the length of $RS$.



**14.** Consider the possible expressions formed when each □ is filled with a + or a − sign in $\square\frac{1}{1}\square\frac{1}{2}\square\frac{1}{3}\square\frac{1}{4}\square\frac{1}{5}\square\frac{1}{6}$. For example, $-\frac{1}{1}+\frac{1}{2}-\frac{1}{3}-\frac{1}{4}+\frac{1}{5}+\frac{1}{6}$ which has value $-\frac{43}{60}$ is one of the $2^6$ possible expressions.

Among these expressions, what is the smallest possible positive value?

A $\frac{1}{60}$ B $\frac{1}{30}$ C $\frac{1}{20}$ D $\frac{1}{10}$ E $\frac{1}{5}$

SOLUTION **C**

It would take too much work to evaluate all the possible expressions, so we aim to find a method that avoids the need to do this.

Calculations with fractions are rather awkward, so our first move is to write all the fractions over a common denominator to turn the question into one about whole numbers.

We have

$$\pm\frac{1}{1}\pm\frac{1}{2}\pm\frac{1}{3}\pm\frac{1}{4}\pm\frac{1}{5}\pm\frac{1}{6}=\frac{\pm60\pm30\pm20\pm15\pm12\pm10}{60}.$$

We now see that we need to find the smallest positive value of the expressions

$$\pm60\pm30\pm20\pm15\pm12\pm10. \qquad (*)$$

This value will be $P-Q$, where $P$ is the sum of the positive terms in (*) and $Q$ is the sum of the terms preceded by a minus sign. So we need to find the values of $P$ and $Q$, with $P>Q$, for which $P-Q$ is as small as possible.

We have $P+Q=60+30+20+15+12+10=147$. Since $P+Q$ is an odd number, the difference $P-Q$ which equals $(P+Q)-2Q$ must also be odd.

To have $P-Q=1$, we need to have $P=74$ and Q = 73. However, it may be seen that there is no way to choose numbers from 60, 30, 20, 15, 12, 10 with sum 74.

To have $P-Q=3$, we need to have $P=75$ and $Q=72$. We could achieve this by having $P=60+15$ and $Q=30+20+12+10$.

Therefore the smallest possible positive value is $\frac{3}{60}=\frac{1}{20}$. This can be obtained as the value of

$$\frac{60-30-20+15-12-10}{60}=+\frac{1}{1}-\frac{1}{2}-\frac{1}{3}+\frac{1}{4}-\frac{1}{5}-\frac{1}{6}.$$

FOR INVESTIGATION

**14.1** What is the smallest possible value of the expression

$$\square\frac{1}{1}\square\frac{1}{2}\square\frac{1}{3}\square\frac{1}{5}\square\frac{1}{6}\square\frac{1}{7}\square\frac{1}{8}.$$

where each box is filled with either a + sign or a − sign?

**15.** The mean of $n$ numbers is $n$, $n > 2$. One of the numbers is $4n$ and another is $n - 6$. The other numbers are all equal to $k$.

Which of these expressions gives $k$ in terms of $n$?

A $n - 3$ B $n - \frac{5}{2}$ C $n - 2$ D $n - \frac{3}{2}$ E $n - 1$

SOLUTION **A**

Since the mean of the $n$ numbers is $n$, their sum is $n \times n$, that is, $n^2$.

Since the $n$ numbers are made up of the numbers $4n$ and $n - 6$ together with $n - 2$ numbers each equal to $k$, their sum is $4n + (n - 6) + (n - 2)k$, that is, $5n - 6 + (n - 2)k$.

These two expressions for the sum of the $n$ numbers must be equal. Therefore

$$n^2 = 5n - 6 + (n - 2)k$$

and hence $(n - 2)k = n^2 - 5n + 6 = (n - 2)(n - 3)$. Since $n - 2 \neq 0$, it follows that

$$k = n - 3.$$

FOR INVESTIGATION

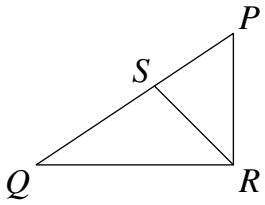
**15.1** The mean of $n$ numbers is $n$, $n > 3$. One of the numbers is $n$, one is $2n - 2$ and one is $2n - 4$. All the other numbers are equal to $k$.

Find the value of $k$ in terms of $n$.

**16.** Triangle $PQR$ has a right angle at $R$. The lengths of $QR$ and $RP$ are $3\sqrt{5}$ and $2\sqrt{5}$ respectively and $\angle QRS = 45°$.

What is the area of triangle $PSR$?

A 5 B 6 C $3\sqrt{5}$ D 7.5 E $7\sqrt{2}$




SOLUTION **B**

Since $\angle PRQ$ is a right angle, and $\angle QRS = 45°$, the line $RS$ bisects $\angle PRQ$. Therefore, by the Angle Bisector Theorem (see Problem 16.1 if you are not familiar with this theorem),

$$\frac{PS}{QS} = \frac{PR}{QR} = \frac{2\sqrt{5}}{3\sqrt{5}} = \frac{2}{3}.$$

The triangles $PSR$ and $QSR$ have bases $PS$ and $QS$, respectively, and the same height, namely the length of the perpendicular from $R$ to $PQ$, say $h$. Hence

$$\frac{\text{area of } PSR}{\text{area of } QSR} = \frac{\frac{1}{2}(PS \times h)}{\frac{1}{2}(QS \times h)} = \frac{PS}{QS} = \frac{2}{3}.$$

It follows that the area of the triangle $PSR$ is two-fifths of the area of the triangle $PRQ$.

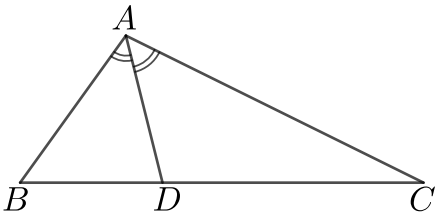
The area of the triangle $PRQ$ is $\frac{1}{2}(QR \times PR) = \frac{1}{2}(3\sqrt{5} \times 2\sqrt{5}) = 15$. Therefore the area of the triangle $PSR$ is given by

$$\frac{2}{5} \times 15 = 6.$$

FOR INVESTIGATION

**16.1** *The Angle Bisector Theorem*

If in the triangle $ABC$, the point $D$ on $BC$ is where the internal bisector of $\angle BAC$ meets $BC$ then $\dfrac{BD}{DC} = \dfrac{BA}{AC}$.




Find a proof of this theorem.

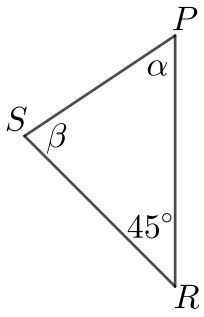
*Note*: This is Proposition 3, Book 6, of Euclid's *Elements.*

**16.2** In the triangle of Question 16, let $\angle SPR = \alpha$ and $\angle RSP = \beta$.

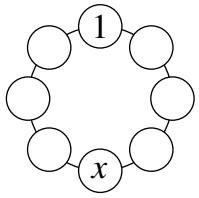
(a) Find the values of $\sin\alpha$ and $\cos\alpha$.

(b) Use the values of $\sin\alpha$ and $\cos\alpha$ to show that $\sin\beta = \dfrac{5}{\sqrt{26}}$.

(c) Use a calculator to check that, to the nearest degree, $\beta$ is 79°.



**17.** The numbers 2, 3, 4, 5, 6, 7 and 8 are to be placed in the empty circles so that (i) there is one number in each circle, and (ii) the sum of the numbers in each pair of adjacent circles is prime.




How many possible values are there for $x$?

A 0 B 1 C 2 D 3 E 4

SOLUTION **C**

When the numbers are placed round the circle, the sum of adjacent numbers will be greater than 2 and has to be a prime. Hence it will be an odd prime. Therefore the numbers round the circle must be alternately odd and even.

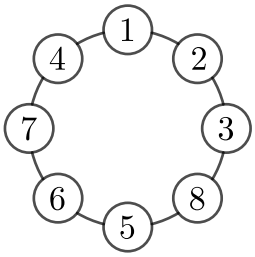
Since 7 + 2 and 7 + 8 are not prime, the number 7 has to be placed between 4 and 6. It follows that 1 cannot be adjacent to both 4 and 6.

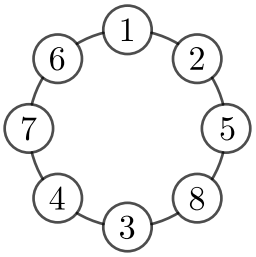
Since 8 + 1 and 8 + 7 are not prime, the number 8 has to be placed between 3 and 5. Hence 1 cannot be adjacent to 8.

It follows that 1 has to be adjacent to 2 and to either 4 or 6.

We consider first the case where 1 is adjacent to 2 and 4. Then, as 7 is between 4 and 6, 4 will be followed by 7 and 7 by 6. 6 cannot be followed by 3 since 6 + 3 = 9 which is not prime. Therefore 6 must be followed by 5. Then as 8 is between 3 and 5, 5 is followed by 8 and 8 by 3.

It follows that the numbers in this case will be arranged round the circle as in the left-hand diagram, or in the same order clockwise instead of anticlockwise, with $x = 5$ in both arrangements.




In the case where 1 is adjacent to 2 and 6, the numbers must be arranged as in the right-hand diagram, or in the same order clockwise instead of anticlockwise, with $x = 3$ in both arrangements.

Therefore there are two possible values for $x$, namely, 3 and 5.

FOR INVESTIGATION

**17.1** Show that if 1 is adjacent to 2 and 6, then the numbers must be arranged round the circle in the order 1, 2, 5, 8, 3, 4, 7, 6, either clockwise or anticlockwise.

**18.** Let $g(x) = x^{\frac{1}{n}}$. Each of $x$, $a$ and $b$ are positive real numbers and $n$ is a positive integer. Which of the following is a simplification of

$$g(a) \times g(b^2) \times g(a^3) \times g(b^4) \times \ldots \times g(a^{2n-1}) \times g(b^{2n})?$$

A $a^n b^{n+1}$ B $a^n b^{n+2}$ C $a^{n+1} b^{n+2}$ D $a^{2n} b^{2n+2}$ E $a^{2n+1} b^{2n+2}$

SOLUTION **A**

We have

$$\begin{aligned} & g(a) \times g(b^2) \times g(a^3) \times g(b^4) \times \cdots \times g(a^{2n-1}) \times g(b^{2n}) \\ = \; & a^{\frac{1}{n}} \times (b^2)^{\frac{1}{n}} \times (a^3)^{\frac{1}{n}} \times (b^4)^{\frac{1}{n}} \times \cdots \times (a^{2n-1})^{\frac{1}{n}} \times (b^{2n})^{\frac{1}{n}} \\ = \; & a^{\frac{1}{n}} \times b^{\frac{2}{n}} \times a^{\frac{3}{n}} \times b^{\frac{4}{n}} \times \cdots \times a^{\frac{2n-1}{n}} \times b^{\frac{2n}{n}} \\ = \; & a^X \times b^Y, \end{aligned}$$

where,

$$X = \frac{1}{n}(1 + 3 + \cdots + (2n-1)) \text{ and } Y = \frac{2}{n}(1 + 2 + \cdots + n).$$

That is,

$$X = \frac{1}{n}(n^2) = n \text{ and } Y = \frac{2}{n}\left(\frac{n(n+1)}{2}\right) = n + 1.$$

It follows that

$$\begin{aligned} & g(a) \times g(b^2) \times g(a^3) \times g(b^4) \times \cdots \times g(a^{2n-1}) \times g(b^{2n}) \\ = \; & a^n b^{n+1}. \end{aligned}$$

FOR INVESTIGATION

**18.1** (a) Prove that for every positive integer $n$,

$$1 + 3 + \cdots + (2n-1) = n^2.$$

(b) Prove that for every positive integer $n$,

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}.$$

[*Note*: The sum, $1 + 2 + \cdots + n$, of the first $n$ positive integers is the $n$th *triangular number* . So $\frac{n(n+1)}{2}$ is the formula for the $n$th triangular number.]

**19.** The largest integer less than or equal to $x$ is represented by $\lfloor x \rfloor$. For example $\lfloor 5.8 \rfloor = 5$ and $\lfloor 3 \rfloor = 3$. For how many positive integers $x$ does $\lfloor x^{\frac{1}{2}} \rfloor + \lfloor x^{\frac{1}{3}} \rfloor = 8$?

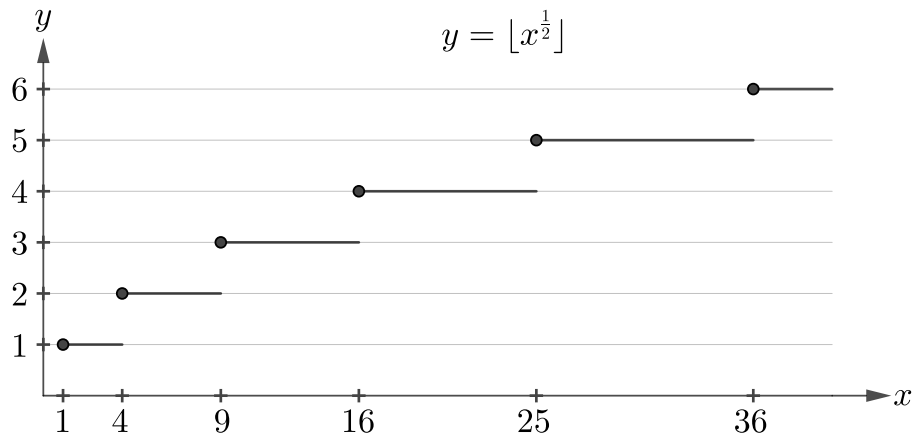
A 9 B 10 C 11 D 12 E 13

SOLUTION **A**

The notation $x^{\frac{1}{2}}$ is used to mean the square root of $x$. This choice is made so that the index rule $x^m \times x^n = x^{m+n}$, holds when $m$ and $n$ are fractions. For this to work we need to have $(x^{\frac{1}{2}})^2 = x^{\frac{1}{2}} \times x^{\frac{1}{2}} = x^{\frac{1}{2}+\frac{1}{2}} = x^1 = x$, so that $x^{\frac{1}{2}}$ has to be the square root of $x$.

It follows that $\lfloor x^{\frac{1}{2}} \rfloor$ is the integer part of the square root of $x$. For example, $\lfloor 4^{\frac{1}{2}} \rfloor = 2$, $\lfloor 5^{\frac{1}{2}} \rfloor = \lfloor 2.236... \rfloor = 2$, $\lfloor 6^{\frac{1}{2}} \rfloor = \lfloor 2.449... \rfloor = 2$, $\lfloor 7^{\frac{1}{2}} \rfloor = \lfloor 2.645... \rfloor = 2$, $\lfloor 8^{\frac{1}{2}} \rfloor = \lfloor 2.828... \rfloor = 2$, and $\lfloor 9^{\frac{1}{2}} \rfloor = \lfloor 3 \rfloor = 3$.

In general, for each positive integer $n$, we have $\lfloor x^{\frac{1}{2}} \rfloor = n$ for $n^2 \leq x < (n+1)^2$. It follows that the graph of $y = \lfloor x^{\frac{1}{2}} \rfloor$ for $x \geq 1$, consists of a sequence of horizontal line segments, with jumps at each positive integer which is a square, as shown below:




The graph of $y = \lfloor x^{\frac{1}{3}} \rfloor$ is similar, with the jumps occurring at integers that are cubes, because the notation $\lfloor x^{\frac{1}{3}} \rfloor$ means the integer part of the cube root of $x$. [See Problem 19.1.]

When $\lfloor x^{\frac{1}{2}} \rfloor \leq 4$, we have $x < 25$. When $x < 25$, we have $\lfloor x^{\frac{1}{3}} \rfloor < 3$, and it follows that $\lfloor x^{\frac{1}{2}} \rfloor + \lfloor x^{\frac{1}{3}} \rfloor < 4 + 3 = 7$. Therefore the equation $\lfloor x^{\frac{1}{2}} \rfloor + \lfloor x^{\frac{1}{3}} \rfloor = 8$ is not true when $\lfloor x^{\frac{1}{2}} \rfloor \leq 4$.

When $\lfloor x^{\frac{1}{2}} \rfloor \geq 6$, we have $x \geq 36$. When $x \geq 36$, we have $\lfloor x^{\frac{1}{3}} \rfloor \geq 3$, and it follows that $\lfloor x^{\frac{1}{2}} \rfloor + \lfloor x^{\frac{1}{3}} \rfloor \geq 6 + 3 = 9$. Therefore the equation $\lfloor x^{\frac{1}{2}} \rfloor + \lfloor x^{\frac{1}{3}} \rfloor = 8$ is not true when $\lfloor x^{\frac{1}{2}} \rfloor \geq 6$.

It follows that the equation $\lfloor x^{\frac{1}{2}} \rfloor + \lfloor x^{\frac{1}{3}} \rfloor = 8$ holds only when $\lfloor x^{\frac{1}{2}} \rfloor = 5$ and $\lfloor x^{\frac{1}{3}} \rfloor = 3$.

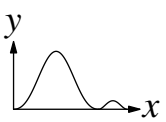
$\lfloor x^{\frac{1}{2}} \rfloor = 5$ for $25 \leq x < 36$ and $\lfloor x^{\frac{1}{3}} \rfloor = 3$ for $27 \leq x < 64$. Hence $\lfloor x^{\frac{1}{2}} \rfloor + \lfloor x^{\frac{1}{3}} \rfloor = 8$ just for $27 \leq x < 36$. Therefore the only positive integer solutions of this equation are $x = 27, 28, 29, 30, 31, 32, 33, 34$ and $35$. Hence the equation has nine positive integer solutions.

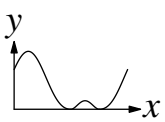
FOR INVESTIGATION

**19.1** Draw the graph of $y = \lfloor x^{\frac{1}{3}} \rfloor$ for $1 \leq x \leq 64$.

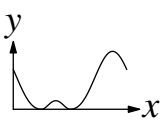
**19.2** Find all the positive integers that are solutions of the equation $\lfloor x^{\frac{1}{2}} \rfloor + \lfloor x^{\frac{1}{3}} \rfloor = 10$.

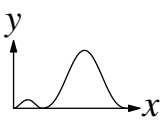
**20.** Which of the following is a sketch of the graph of $y = 1 + \sin x \cos x - \sin x - \cos x$ for $0° \leq x° \leq 360°$?

A 


B 


C


D 


E 


SOLUTION **E**

In order to answer this question you don't need to sketch the graph of $y = 1 + \sin x \cos x - \sin x - \cos x$ for $0° \leq x° \leq 360°$.

In context of the SMC it is safe to assume that one of the options gives a correct sketch of the graph. Therefore, all you need to do is to use some of the values of $\sin x$ and $\cos x$ that you know to eliminate four of the five given options.

We have $\sin 0° = 0$ and $\cos 0° = 1$.

Therefore when $x° = 0°$, the value of $y$ is given by $y = 1 + 0 \times 1 - 0 - 1 = 1 + 0 + 0 - 1 = 0$. Hence the origin lies on the graph.

This rules out the graphs of options B and D.

Similarly, $\sin 90° = 1$ and $\cos 90° = 0$.

Therefore when $x° = 90°$, the value of $y$ is given by $y = 1 + 1 \times 0 - 1 - 0 = 1 + 0 - 1 - 0 = 0$.

Therefore the point with coordinates $(90, 0)$ lies on the graph.

This rules out the graphs of options A and C.

We conclude that it is option E that gives the correct sketch.

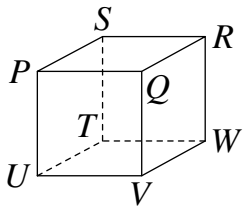
FOR INVESTIGATION

**20.1** Find all the solutions of the equation

$$1 + \sin x \cos x - \sin x - \cos x = 0$$

for which $0° \leq x° \leq 360°$.

**21.** The diagram shows a clear plastic cube, $PQRSTUVW$. An ant starts at vertex $S$ and walks on the surface of the solid cube. It travels along straight line segments (first to $Q$, $U$ or $W$), alternately across the diagonal of a face and then along an edge. An example of a possible five-segment journey is $SQRTSW$.

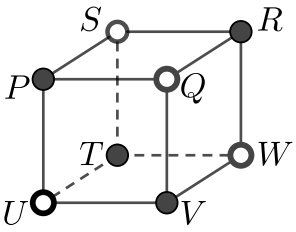



At the end of a journey of 81 segments, which of the following vertices is it impossible for the ant to reach?

A $S$ B $Q$ C $U$ D $W$ E $V$

SOLUTION **E**

We colour the vertices $P$, $R$, $T$ and $V$ black, and the vertices $Q$, $S$, $U$ and $W$ white, as shown in the diagram.




Then two vertices diagonally opposite each other on a face are always the same colour, whereas two vertices at the opposite ends of an edge are always different colours.

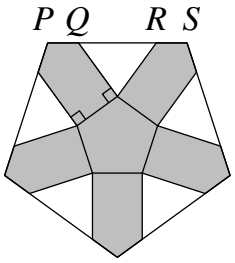
The ant starts at the white vertex $S$. In a journey of 81 segments the ant travels along a diagonal 41 times and along an edge 40 times. So the vertex it ends up at after each segment changes colour 40 times. Hence the ant ends up at a vertex of the same colour as the vertex where it started, that is, a white vertex. Therefore it is impossible for the ant to reach the black vertex $V$ after a journey of 81 segments.

[*Note* The above argument proves that the ant cannot end up at a black vertex, but we have not shown that it is possible for it to arrive at each of the white vertices. So, for a complete answer you would need to show that it is possible for the ant to arrive at each of the white vertices $S$, $Q$, $U$ and $W$ given as the other options. See Problem 21.1.]

FOR INVESTIGATION

**21.1** Show that it is possible for the ant to arrive at any of the white vertices $Q$, $S$, $U$ and $W$, after a journey of 81 segments.

**22.** As shown in the diagram, a playground is built in the shape of a regular pentagon. In the centre is a smaller regular pentagon, with lines perpendicular to each side at each vertex of the smaller pentagon. The order of the rotational symmetry of the diagram is 5. The side-length of the larger pentagon is 10 m, and $PQ = RS = 2$ m.

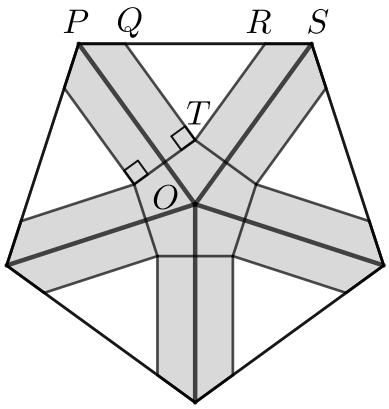



What fraction of the area of the playground is the area of the shaded region?

A $\frac{3}{5}$ B $\frac{5}{8}$ C $\frac{9}{25}$ D $\frac{16}{25}$ E $\frac{81}{125}$

SOLUTION **D**

We let $O$ be the centre of the two regular pentagons and let $T$ be the point shown in the diagram on the right.




Because of the symmetry of the figure, $PO = SO$ and $\angle POS = \frac{1}{5}(360°) = 72°$.

The interior angles of a regular pentagon are 108°. The angles at $T$ have sum 360°. Therefore $\angle QTR = 360° - 90° - 90° - 108° = 72°$. Also, by the symmetry of the figure, $QT = RT$.

Therefore $POS$ and $QTR$ are both isosceles triangles with apex angle 72°. Hence these triangles are similar.

The ratio of the areas of similar triangles equals the ratio of the squares of the lengths of their corresponding sides. [You are asked to prove this in Problem 22.1.]

$PS = 10$ m, and $QR = PS - PQ - RS = (10 - 2 - 2)$ m $= 6$ m.

Therefore

$$\text{area of } QRT : \text{area of } POS = 6^2 : 10^2 = 36 : 100 = 9 : 25.$$

By the symmetry of the figure it follows that the total area of the unshaded triangles is $\frac{9}{25}$th of the area of the playground. Hence the area of shaded region is $\left(1 - \frac{9}{25}\right)$th $= \frac{16}{25}$th of the area of the playground.

FOR INVESTIGATION

**22.1** Prove that the ratio of the areas of similar triangles equals the ratio of the squares of the lengths of their corresponding sides.

**22.2** Prove that each interior angle of a regular pentagon is 108°.

**22.3** Show that, in the diagram of Question 22,

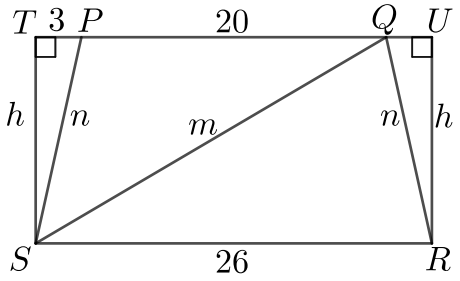
$$\frac{\text{area of the smaller regular pentagon}}{\text{area of the larger regular pentagon}} = \frac{4\sin^2(54°)}{25}.$$

**23.** Let $PQRS$ be a trapezium with $QP$ parallel to $RS$ and with $QP = 20$, $RS = 26$ and $PS = QR = n$, for some positive integer $n$. The length of the diagonal $QS$ is also a positive integer.

What is the maximum possible value of $n$?

SOLUTION **129**

We let $T$ and $U$ be the points where the lines through $S$ and $R$, respectively, perpendicular to $PQ$ meet the extension of $PQ$.




$TURS$ is a rectangle, and hence $ST = RU = h$, say.

Since $PS = QR$, it follows that the right-angled triangles $TPS$ and $URQ$ are congruent. [You are asked to show this in Problem 23.1.]

It follows that $TP = QU = \frac{1}{2}(TU - PQ) = \frac{1}{2}(RS - PQ) = \frac{1}{2}(26 - 20) = 3$.

Let $QS = m$. By Pythagoras' Theorem, applied to the right-angled triangles $QTS$ and $PTS$, we have $m^2 = h^2 + 23^2$ and $h^2 = n^2 - 3^2$. It follows that $m^2 = (n^2 - 3^2) + 23^2 = n^2 - 9 + 529$. Therefore $m^2 = n^2 + 520$. Hence $m^2 - n^2 = 520$. Therefore

$$(m - n)(m + n) = 520. \qquad (*)$$

We now seek the solution of equation (*) in which $m$ and $n$ are both positive integers, and $n$ has its largest possible value.

We note first that $(m - n) + (m + n) = 2m$, which even. Hence $m - n$ and $m + n$ are both even or both odd. Since their product is the even number 520, they are both even.

Therefore we need only consider the pairs of positive even numbers, $m - n$ and $m + n$ whose product is 520, and the corresponding values of $m$ and $n$. These are shown in the following table.

| $m - n$ | $m + n$ | $m$ | $n$ |
|---|---|---|---|
| 2 | 260 | 131 | 129 |
| 4 | 130 | 67 | 63 |
| 10 | 52 | 31 | 21 |
| 20 | 26 | 23 | 3 |

From this table we see that the maximum possible value of $n$ is 129.

FOR INVESTIGATION

**23.1** Explain why the triangles $TPS$ and $URQ$ are congruent.

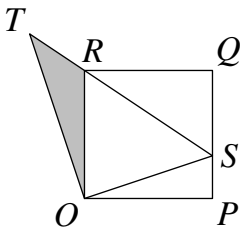
**23.2** (a) Suppose that in the diagram of the above solution $PQ = a$ and $RS = b$. Find an equation which relates the values of $a$, $b$, $m$ and $n$.

(b) Find all the possible positive integer values for $n$ in the case where $a = 9$ and $b = 40$.

(c) Show that when $a$ and $b$ are both even, the largest possible value of $n$ is $\dfrac{ab}{4} - 1$.

**24.** $OPQR$ is a square. The point $S$ lies on $PQ$ such that $PS : SQ = 1 : 2$. Points $S$, $R$ and $T$ lie on a straight line and angle $TOS$ is a right angle. The area of the triangle $ORT$ is 30.

What is the area of $OPQR$?




SOLUTION **140**

Our strategy is to calculate the area of the triangle $ORT$ in terms of the side-length of the square $OPQR$. We then use the given area of $ORT$ to work out the side-length of the square. This then enables us to calculate the area of the square.

We can take the side of the square $OR$ as the base of the triangle $ORT$. We next need to find the height of the triangle in terms of the length of $OR$.

One method for doing this is to use coordinate geometry. The question setter may have chosen '$O$' as the label for one vertex of the square, as a hint to use this method. Problem 23.1 leads you through a solution using coordinate geometry. Our method uses similar triangles.

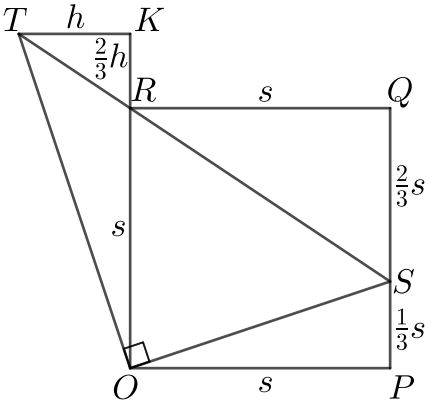
The height of the triangle $ORT$ with base $OR$ is the length of the perpendicular line from the point $T$ to the extension of $OR$. When this line is added to the diagram this creates two pairs of similar triangles. We use these to calculate the required length.

We let $s$ be the side-length of the square $OPQR$.

Since $PS : SQ = 1 : 2$, the length of $PS$ is $\frac{1}{3}s$ and the length of $SQ$ is $\frac{2}{3}s$.

Let the line through $T$ parallel to $RQ$ meet the line $OR$ extended at the point $K$. We let $h$ be the length of $TK$.

Since $TK$ is parallel to $RQ$, it follows that $\angle KTR = \angle QRS$ and $\angle TKR = \angle RQS = 90°$. Therefore the triangles $KTR$ and $QRS$ are similar.




Hence $\dfrac{KR}{TK} = \dfrac{SQ}{RQ} = \dfrac{\frac{2}{3}s}{s} = \dfrac{2}{3}$.

Therefore $KR = \frac{2}{3}TK = \frac{2}{3}h$.

Now, $\angle ROT + \angle SOR = 90° = \angle POS + \angle SOR$. It follows that $\angle ROT = \angle POS$. Therefore the right-angled triangles $OKT$ and $OPS$ are similar.

Hence $\dfrac{TK}{OK} = \dfrac{SP}{OP}$.

It follows that

$$\frac{h}{s + \frac{2}{3}h} = \frac{\frac{1}{3}s}{s} = \frac{1}{3}.$$

This gives $3h = s + \frac{2}{3}h$. Hence $\frac{7}{3}h = s$. Therefore $h = \frac{3}{7}s$.

The triangle $ORT$ has base $OR$ which has length $s$ and height $TK$ which has length $h$, that is, $\frac{3}{7}s$. Hence the area of this triangle is $\frac{1}{2}(s \times \frac{3}{7}s) = \frac{3}{14}s^2$. Hence, by the information given in the question, $\frac{3}{14}s^2 = 30$. It follows that

$$s^2 = \frac{14}{3} \times 30 = 140.$$

We conclude that the square $OPQR$ has area 140.

FOR INVESTIGATION

**24.1** Let $s$ be the side-length of the square $OPQR$.

We choose coordinate axes with the point $O$ as the origin, the $x$-axis in the direction of $OP$ and the $y$-axis in the direction of $OR$.

(a) What are the coordinates of the points $R$ and $S$?

(b) What is the gradient of the line $OS$?

(c) Find the gradient of the line $OT$ which is perpendicular to $OS$.

(d) Find the equation of the line $OT$.

(e) Find the equation of the line $RS$.

(f) Use the equations of the lines $OT$ and $RS$ to find the coordinates of point $T$ where these lines meet.

(g) Use the coordinates of $T$ to find the height of the triangle $ORT$ in terms of $s$.

(h) Use the given fact that the triangle $ORT$ has area 30 to determine the value of $s^2$. This gives the area of the square $OPQR$.

**24.2** Suppose that in the diagram of this question, $PS : SQ = k : 1 - k$, where $0 \leq k \leq 1$.

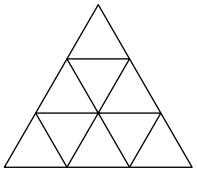
(a) Find the ratio

$$\text{area of the triangle } ORT : \text{area of the square } OPQR$$

in terms of $k$.

(b) Check that your formula for the ratio gives the ratio as $3 : 14$ when $k = \frac{1}{3}$, as in Question 24.

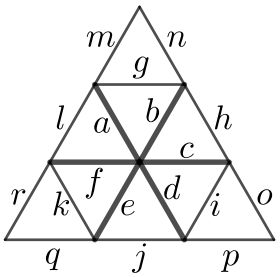
**25.** This diagram, built from equilateral triangles of side-length 1, has a total of 18 edges of length 1. Each edge is to be coloured red, white or blue. Note that different colourings of the edges of the diagram which can be made the same by rotation or reflection of the diagram are to be considered as different colourings.



How many different coloured diagrams can be made, given that each equilateral triangle of side-length 1 must have edges of all three colours?

SOLUTION **528**

In the diagram on the right we have labelled the edges so that we can refer to them.




Suppose that the six edges $a$, $b$, $c$, $d$, $e$ and $f$ that radiate from the centre of the large triangle have been coloured, with no two edges of the same triangle being the same colour.

We begin by considering the number of ways in which the remaining edges could be coloured.

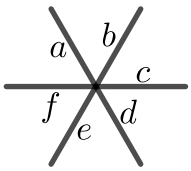
Since the edges $a$ and $b$ use two of the colours red, white and blue, the edge $g$ must use the third colour. For similar reasons there is just one possibility for the colours of each of the edges $h$, $i$, $j$, $k$ and $l$.

The edges $m$ and $n$ must use the two colours not used by $g$. These two colours could be either way round. For example, if $g$ is red, $m$ could be white and $n$ blue, or $m$ could be blue with $n$ white. Thus once a colour has be chosen for $g$, there are two ways in which $m$ and $n$ could be coloured.

Likewise, once colours have been chosen for the edges $i$ and $k$, there are two choices for the colours for $o$ and $p$, and two choices for $q$ and $r$.

It follows that once colours have been chosen for $a$, $b$, $c$, $d$, $e$ and $f$, there are $2 \times 2 \times 2 = 8$ ways in which all the other edges can be coloured.

Hence there are $8N$ different coloured diagrams that can be made, where $N$ is the number of ways of colouring the edges $a$, $b$, $c$, $d$, $e$ and $f$, so that adjacent edges are given different colours.




We count the number of ways of colouring the edges $a$, $b$, $c$, $d$, $e$ and $f$, according to the number of different colours used to colour the edges $a$, $c$ and $e$.

1 colour: There are three choices for this one colour. Once this colour has been chosen for $a$, $c$ and $e$, there are two possible colours for each of the edges $b$, $d$ and $f$. Hence there are $3 \times (2 \times 2 \times 2) = 24$ colourings of this type.

2 colours: In this case one of the edges $a$, $c$ and $e$ will be one colour and the other two of them will be the second colour. The one edge can be chosen in 3 ways and its colour in 3 ways. The second colour can then be chosen in 2 ways. So there are $3 \times 3 \times 2 = 18$ ways to colour $a$, $c$ and $e$ using two colours.

In each of these colourings one of the edges $b$, $d$ and $f$ will be between edges that have the same

colour. There are two possible colours for this edge. The other two of $b$, $d$ and $f$ will be between two edges having different colours. The third colour is the only possible choice for the colour of these edges.

So, overall, there are $18 \times 2 \times 1 \times 1 = 36$ colourings of this type.

3 colours: There are $3 \times 2 \times 1 = 6$ ways in which each of the colours red, white and blue may be assigned to the edges $a$, $c$ and $e$. In this case each of the edges $b$, $d$ and $f$ is between edges of different colours, and so must have the colour not used by the edges which it lies between.

So there are 6 colourings of this type.

We now see that $N = 24 + 36 + 6 = 66$. Hence there are $8 \times 66 = 528$ colourings of the diagram that meet the required conditions.

### FOR INVESTIGATION

**25.1** (a) Suppose that there are four colours, red, green, blue and yellow, which may be used to colour the 18 edges of the triangles in the diagram of this question.

In this case, how many different coloured diagrams can be made, given that each equilateral triangle of side-length 1 must have edges whose colours are all different?

(b) Suppose that there are $K$ colours which may be used to colour the 18 edges of the triangles in the diagram of this question.

Find a formula, in terms of $K$, for the number of different coloured diagrams that can be made, given that each equilateral triangle of side-length 1 must have edges whose colours are all different.